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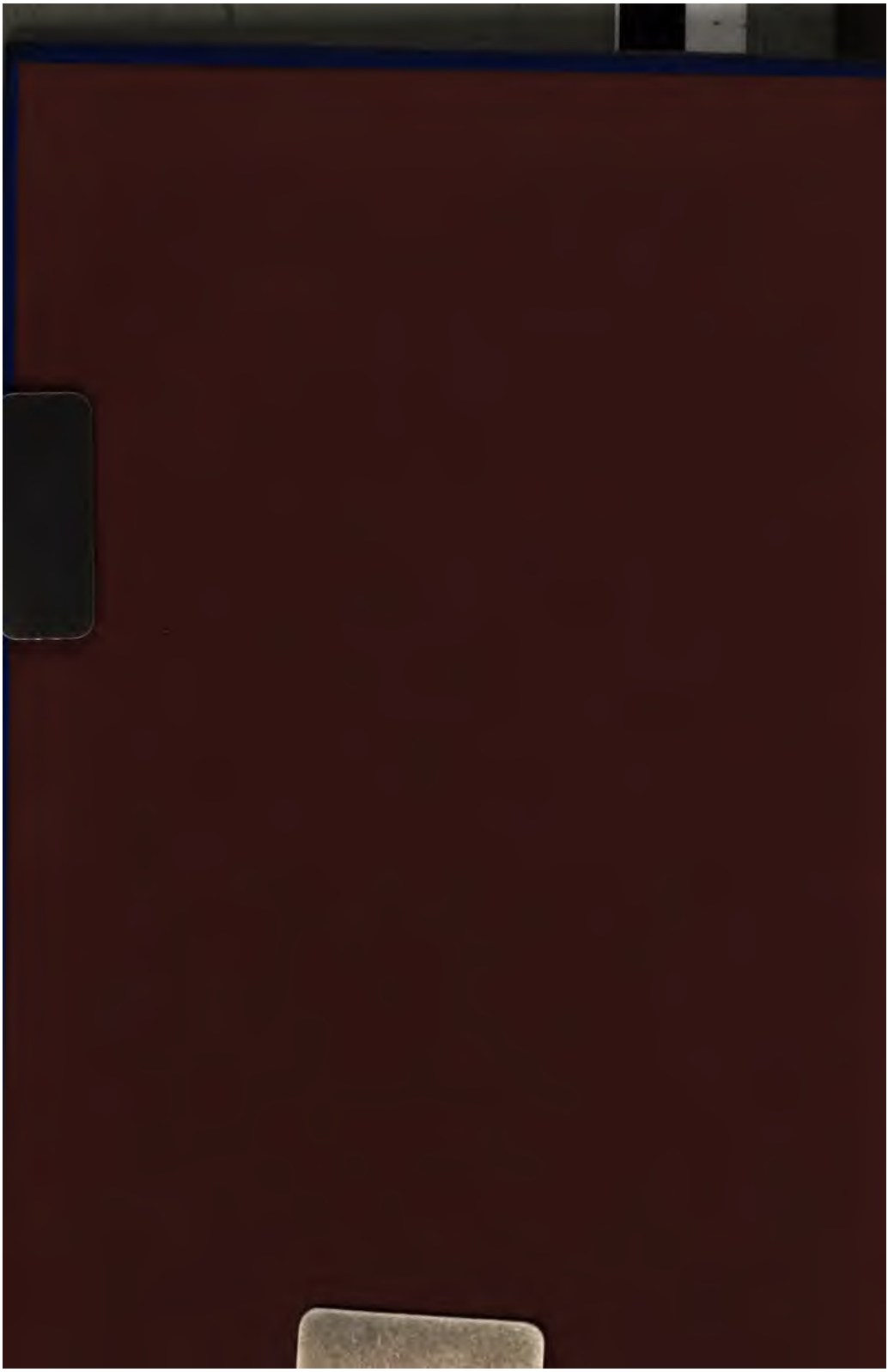

THE THEORY OF NAVIGATION

AND

NAUTICAL ASTRONOMY.

THE

THEORY OF NAVIGATION

AND

NAUTICAL ASTRONOMY,

TOGETHER WITH THE ELEMENTS OF

PLANE AND SPHERICAL TRIGONOMETRY,

WITH EXAMPLES FOR THE

USE OF MARINE CADETS.

BY

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INTRODUCTION.

THE following pages contain the proofs of the principal Formulæ of Plane and Spherical Trigonometry, and the Theory and "Investigations" of the various Problems of Nautical Astronomy in general use.

The Formulæ are so divided into Classes, and numbered, that they may be easily referred to. At the end of the work are Exercises in the Solutions of all cases of Plane and Spherical Triangles, together with Questions (some of them analytical) depending upon a ready knowledge of the various Formulæ previously proved. These Questions, with the exception of a very few, are *original* in their nature; but they are not very numerous, as the work is, mainly, and in pretension, theoretical.

Some of the Problems, worked out as Examples, are well known amongst Mathematicians, and are to be found in all standard works on Navigation and Nautical Astronomy. For a few Investigations in the latter science, I am indebted to sources acknowledged where those methods are inserted.

As a representation of the course of Theory, taught on board the "Worcester" Training Ship, where, I may add, the want of such a work has been for a long time much felt, it is hoped, that, apart from the interests of those for whom it has been compiled, it may prove of use, generally, to others, who are making Navigation and Nautical Astronomy their chief objects of study.

W. T. READ.

H.M.S. "Worcester,"
January, 1869.

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PLANE AND SPHERICAL

TRIGONOMETRY.

C

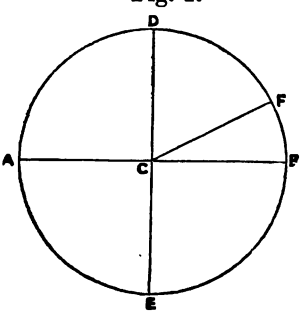
PLANE TRIGONOMETRY.

The literal meaning of the word Trigonometry is, the "measure of a triangle." The science therefore supplies us with certain rules and formulæ from which the measurements of the sides and angles of plane and spherical triangles are determined.

In a plane triangle, the sides and angles are of different kinds with respect to measurement, and therefore the relation between them, and the sizes of the angles, can only be discovered by comparing the sides with certain lines in a circle, on which lines the arcs of the circle which measure the angles, depend.

If the circumference of a circle be divided into 360 equal parts, each part is called a degree, (marked thus °); the 60th part of a degree is, a minute (′); and that of a minute, a second (″).

Fig. 1.



Let DAEB be a circle, and let two of its diameters, DE, and AB, be at right angles to one another. In the circumference DB, take a point F, and join CF.

Then, since the circumference of a circle contains 360°, the quarter of it, or a quadrant, contains 90°; therefore DB contains 90°. And since the angles round any point, are equal to four right angles, therefore the right angle DCB, likewise contains 90°; hence DB and the right angle DCB, measure the same; and we have this proportion,

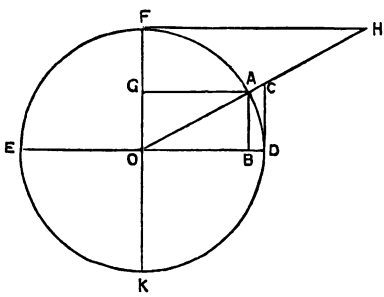
As DB: FB :: ∠DCB: ∠FCB.
But DB measures DCB
Therefore FB measures FCB.

Therefore any angle at the centre of a circle, contains the same number of degrees, as the arc subtending it.

Defns Two arcs, whose sum is a quarter of a circle, are called **complements** of each other, thus FB is the complement of FD, and *vice versâ*

Two arcs, whose sum is a semi-circle, are called **supplements** of each other, as FB and FA, in the figure.

Fig. 2.



Let FDKE be a circle, and FK and ED, two diameters at right angles to one another. In the circumference FD take any point A. From A, draw AB perpendicular to ED. Join OA and produce it indefinitely. From A draw AG perpendicular to FK, and from D and F, draw DC, and FH perpendicular to OD and OF, meeting OA produced in C, and H. Then, with respect to the arc AD, or the angle AOD,

AB is called the **Sine.**
OB „ „ **Cosine.**
CD „ „ **Tangent.**
OC „ „ **Secant.**
FH „ „ **Cotangent.**
OH „ „ **Cosecant.**
BD „ „ **Versed Sine.**
EB „ „ **Suversed Sine.**

In like manner AG will be the sine of the arc AF; OG its cosine; FH its tangent; CD its cotangent, &c.

Since FA and AD are complements of each other, and since AB is equal to OG, and AG to OB, we have

AB, which is the sine of AD = GO, which is the cosine of AF,
OB, which is the cosine of AD = AG, which is the sine of AF,
CD, which is the tangent of AD, is the cotangent of AF,
FH, which is the cotangent of AD is the tangent of AF. Hence the following:

The sine of an angle is equal to the cosine of its complement; the cosine to the sine of its complement; the tangent to the cotangent of its complement.

1. It is also manifest from this figure, that if the point A descends to D, the arc AD is 0, and then,

The Sine AB = 0
„ Cosine OB = radius
„ Tangent CD = 0
„ Cotangent FH is infinite
„ Secant OC = radius
„ Cosecant OH is infinite
„ Versed-sine BD = 0
„ Suversed-sine EB = twice the radius.

2. Also, if the point A be in the centre of FD, then AD = 45°, and it is easily seen that

The Sine = the cosine
„ Tangent = cotangent
„ Secant = cosecant

For the points C and H coincide.

3. If the point A travel to F, then the arc AD becomes 90°, and its

Sine = radius
Cosine = 0
Tangent is infinite
Cotangent = 0
Secant is infinite
Cosecant = radius.

Algebraical Signs of Functions.

It is conventional amongst mathematicians, to speak of the four quadrants of the circle in fig. 1, as the 1st, 2nd, 3rd, and 4th. It has also been agreed that all sines measured upwards shall be + and downwards — ; *i. e.*, above or below AB, and all cosines right-handed, +, and left-handed, —, *i. e.*, to the right or left of DE.

Hence, in the 1st quadrant, DB, we have sines and cosines both +

In the 2nd quadrant, AD, we have sines, +, cosines, —

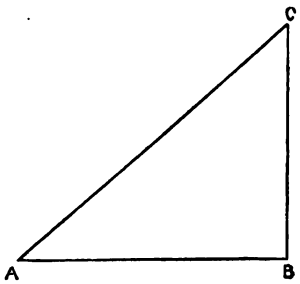
In the 3rd quadrant, AE, we have sines —, cosines —

In the 4th quadrant, EB, we have sines —, cosines +

It can be shown by Class II that the signs of any other functions, such as tangents and secants, may be determined from the above.

The Right-angled Triangle.

Fig. 3.



We have hitherto spoken of the sine, cosine, of an angle, as lines; but in practice they are regarded not as lines, but as quantities; they are, in fact, fractions.

In the triangle ABC, if we take the sides two and two in order, we can form six fractions from them, viz.:

$$\frac{AB}{AC}, \frac{BC}{AC}, \frac{AB}{BC}, \frac{AC}{BC}, \frac{BC}{AB}, \frac{AC}{AB}.$$

Referring to fig. 2, and in it the triangle ABO, if we suppose the radius to be unity or 1, then instead of saying that $\frac{AB}{AO}$ is the sine, we may write down $\frac{AB}{AO} = \frac{AB}{\text{radius}} = \frac{AB}{1}$ as the sine; and, in like manner, $\frac{OB}{OA} = \frac{OB}{1}$ would be the cosine of the angle AOB.

This may be better understood thus. If we suppose CDO to be any triangle, and ABO similar to it, we have, by (Euc. vi. 4),

$$CD : CO :: BA : AO$$

or perp : hyp :: sine O : radius

then, if rad : = 1

$$\frac{\text{perp}}{\text{hyp}} = \text{sine} : \text{ or } \frac{CD}{CO} = \sin O.$$

Also,

$$OD : OC :: OB : OA$$

or base : hyp :: cosine : radius

whence,

$$\frac{\text{base}}{\text{hyp}} = \text{cosine}; \text{ or } \frac{OD}{OC} = \cos O.$$

Again, in the similar triangles, CDO and ABO,

$$AB : BO :: CD : DO$$

or perp : base :: tangent : radius

whence,

$$\frac{\text{perp}}{\text{base}} = \text{tangent} : \text{ or } \frac{AB}{BO} = \tan O.$$

and so on for the other three functions, cot, sec, cosec.

If we carry this idea to the right angled triangle ABC, we get the six ratios on which the whole of the formulæ of Trigonometry depend, and which should be thoroughly learnt before proceeding any farther; we have

$$\text{Sine A} = \frac{BC}{AC}: \quad \text{Cosine A} = \frac{AB}{AC}$$

$$\text{Tangent A} = \frac{BC}{AB}: \text{Cotangent A} = \frac{AB}{BC}$$

$$\text{Secant A} = \frac{AC}{AB}: \quad \text{Cosecant A} = \frac{AC}{CB}$$

These six fractions are **proper, improper, or variable.** AC must always be the longest of the three sides, hence $\frac{BC}{AC}$ and $\frac{AB}{AC}$ are represented as $\frac{\text{less}}{\text{greater}}$; in other words, the **sine** and **cosine** of an acute angle, are always **proper** fractions.

The fractions $\frac{AC}{AB}$ and $\frac{AC}{BC}$ are represented as $\frac{\text{greater}}{\text{less}}$; hence the **secant** and **cosecant** are always **improper** fractions. And since AB and BC may be equal or unequal, it follows that $\frac{BC}{AB}$ or $\frac{AB}{BC}$, *i. e.*, the tangent and cotangent, are changeable, they may be **proper** or **improper** fractions. If the angle A exceeds 45°, it is easily seen that the tangent will be **improper**, and the cotangent, **proper**; if less than 45°, the reverse will be the case.

FORMULÆ.

CLASS I.

By Prop. 47 I Euc., we know, that,

$$AC^2 = AB^2 + BC^2$$

Dividing both sides by AC^2,

$$\frac{AC^2}{AC^2} = \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2}$$

(1) or $1 = \text{Cos}^2 A + \text{sin}^2 A$

Again dividing both sides by AB^2

$$\frac{AC^2}{AB^2} = \frac{AB^2}{AB^2} + \frac{BC^2}{AB^2}$$

(2) $\text{Sec}^2 A = 1 + \tan^2 A$

And dividing by BC^2

$$\frac{AC^2}{BC^2} = \frac{AB^2}{BC^2} + \frac{BC^2}{BC^2}.$$

(3) $\text{Cosec}^2 A = \text{Cot}^2 A + 1.$

From (1) come the following:—

Since (1) $\sin^2 A + \cos^2 A = 1.$

$\therefore \sin^2 A = 1 - \cos^2 A$

and $\sin A = \sqrt{1 - \cos^2 A}$

also $\cos^2 A = 1 - \sin^2 A$

and $\cos A = \sqrt{1 - \sin^2 A}$

From (2) come the following:

Since (2) $\text{Sec}^2 A = 1 + \tan^2 A$

$\therefore \sec A = \sqrt{1 + \tan^2 A}$

and $\tan^2 A = \sec^2 A - 1$

„ $\tan A = \sqrt{\sec^2 A - 1}.$

From (3) we have the following:

Since (3) $\text{Cosec}^2 A = 1 + \cot^2 A$

$\therefore \text{cosec } A = \sqrt{1 + \cot^2 A}$

and $\cot^2 A = \text{cosec}^2 A - 1.$

„ $\cot A = \sqrt{\text{cosec}^2 A - 1}.$

By referring to fig. 2, we have,

$$BD = DO - OB$$

(4) $\text{Vers } A = \text{rad} - \cos A$

and $EB = EO + OB$

(5) or $\text{Suvers } A = \text{rad} + \cos A$

Now, if radius be supposed to be 1, we have,

(6) $\text{Vers } A \text{ suvers } A = (1 + \cos A)(1 - \cos A)$

$= 1 - \cos^2 A = \sin^2 A$

CLASS II.

In the right angled triangle, ABC,

$$\text{Tan A} = \frac{\text{BC}}{\text{AB}}$$

Divide numerator and denominator by AC

$$\text{(1) Then Tan A} = \frac{\frac{\text{BC}}{\text{AC}}}{\frac{\text{AB}}{\text{AC}}} = \frac{\text{Sin A}}{\text{Cos A}}$$

$$\text{Again, Cot A} = \frac{\text{AB}}{\text{BC}}$$

Divide numerator and denominator by AC

$$\text{(2) Then Cot A} = \frac{\frac{\text{AB}}{\text{AC}}}{\frac{\text{BC}}{\text{AC}}} = \frac{\text{Cos A}}{\text{Sin A}}$$

In the same way the following are proved:

$$\text{(3)} \quad \text{Tan A} = \frac{\text{BC}}{\text{AB}} = \frac{\frac{\text{BC}}{\text{BC}}}{\frac{\text{AB}}{\text{BC}}} = \frac{1}{\cot \text{A}}\text{: hence tan A . cot A} = 1$$

$$\text{(4)} \quad \text{Sec A} = \frac{\text{AC}}{\text{AB}} = \frac{\frac{\text{AC}}{\text{AC}}}{\frac{\text{AB}}{\text{AC}}} = \frac{1}{\cos \text{A}}\text{: hence sec A . cos A} = 1.$$

$$\text{(5)} \quad \text{Cosec A} = \frac{\text{AC}}{\text{BC}} = \frac{\frac{\text{AC}}{\text{AC}}}{\frac{\text{BC}}{\text{AC}}} = \frac{1}{\sin \text{A}}\text{: hence cosec A . sin A} = 1.$$

CLASS III.

Since in the triangle ABC, the angle $\text{B} = 90°$ $\therefore$ $\text{A} + \text{C} = 90°$ (I. Euc. 32) $\therefore$ $\text{C} = (90° - \text{A})$

$$\text{And sin A} = \frac{\text{BC}}{\text{AC}} = \cos \text{C} = \cos (90° - \text{A})$$

$$\text{Also } \cos A = \frac{AB}{AC} = \sin C = \sin (90° - A)$$

$$\text{,, } \tan A = \frac{BC}{AB} = \cot C = \cot (90° - A)$$

$$\text{,, } \cot A = \frac{AB}{BC} = \tan C = \tan (90° - A)$$

$$\text{,, } \sec A = \frac{AC}{AB} = \operatorname{cosec} C = \operatorname{cosec} (90° - A)$$

$$\text{,, } \operatorname{cosec} A = \frac{AC}{BC} = \sec C = \sec (90 - A)$$

These formulæ verify the remarks upon the circle (fig. 2), viz., that "the sine of an angle is equal to the cosine of its complement, &c.

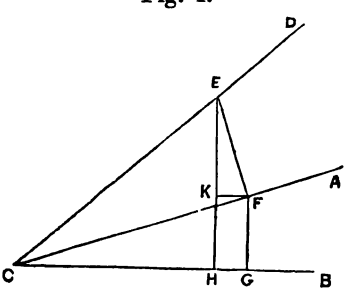
CLASS IV.

Compound Angles.

To prove

(1) $\text{Sin } \overline{A + B} = \sin A \,.\, \cos B + \cos A \sin B$
(2) $\text{Cos } (A + B) = \cos A \cos B - \sin A \sin B$
(3) $\text{Sin } (A - B) = \sin A \,.\, \cos B - \cos A \sin B$
(4) $\text{Cos } (A - B) = \cos A \,.\, \cos B + \sin A \sin B$

Fig. 4.



Let the angle ACB = A, (*i.e.*, contain A°) and the angle ACD = B; then the whole angle DCB = (A + B).

Construction.

In CD take a point E: from E draw EH perpendicular to CB, and EF perpendicular to CA. From F draw FG perpendicular to CB, and FK parallel to CB.

Proof.

To prove that the angle KEF = ACB = A°

$\because$ KF is parallel to CG

$\therefore \angle$ KFC $= \angle$ FCG

Again, $\because$ EFC = 90°

$\therefore$ EFK + KFC = 90°

Also, $\because$ EKF = 90°

$\therefore$ KEF + KFE = 90° (Euc. I − 32)

$\therefore$ KEF + KFE = EFK + KFC

Omit EFK

$\therefore$ KEF = KFC = A

$$\text{Sin (A + B)} = \frac{HE}{EC} = \frac{HK + KE}{EC} = \frac{FG + KE}{EC}$$

$$\frac{FG}{EC} + \frac{KE}{EC}.$$

Multiply numr and denr of the 1st fraction by CF

and of the 2nd „ „ FE

Then,

$$\text{Sin (A + B)} = \frac{FG \cdot FC}{FC \cdot CE} + \frac{KE \cdot EF}{EF \cdot EC}$$

or Sin (A + B) = sin A : cos B + cos A . sin B

Again,

$$\text{Cos (A + B)} = \frac{CH}{CE} = \frac{CG - GH}{CE} = \frac{CG - KF}{CE} = \frac{CG}{CE} - \frac{KF}{CE}$$

Multiply numr and denr of the 1st fraction by CF

and of the 2nd „ „ FE

Then,

$$\text{Cos (A + B)} = \frac{CG \cdot CF}{CF \cdot CE} - \frac{FK \cdot FE}{FE \cdot CE}$$

$$= \text{Cos A . cos B} - \text{sin A sin B}$$

To prove the value of sin $\overline{A - B}$.

Since Sin A = cos (90 − A)

For A write $\overline{A - B}$

Then,

$\text{Sin}(A-B) = \cos(90-\overline{A-B})$

$= \cos\overline{(90-A+B)}$

$= \cos(90-A)\cos B - \sin(90-A)\sin B$

$\therefore \text{Sin}(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

To prove the value of Cos (A−B)

Since $\text{Cos } A = \sin(90 - A)$

For A write $\overline{A - B}$)

Then,

$\text{Cos}(A-B) = \sin(90-\overline{A-B})$

$= \sin\overline{(90-A+B)}$

$= \sin(90-A)\cos B + \cos(90-A)\sin B$

$\therefore \text{Cos}(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

By means of these four formulæ, the values of several compound angles may be determined.

(5) To prove $\sin(180 - A) = \sin A$

(6) and $\cos(180 - A) = -\cos A$

Expanding first by formula (3)

Since $\sin(B - A) = \sin B \cos A - \cos B \cdot \sin A$

For B write 180°.

Then $\sin(180 - A) = \sin 180 \cos A - \cos 180 \sin A$

But by referring to the circle, it will be seen that $\sin 180° = 0$, and $\cos 180° = -1$

Therefore $\sin(180 - A) = 0 \times \cos A - (-1)\sin A$

or $\sin(180 - A) = \sin A$

Again, expanding by (4)

$\text{Cos}(B - A) = \cos B \cos A + \sin B \sin A$

For B write 180°

Then $\cos(180 - A) = \cos 180 \cos A + \sin 180 \sin A$

$= -1 \times \cos A + 0 \times \sin A$

or $\cos(180 - A) = -\cos A$

(7) Find a value for $\sin(270 - A)$

By (3), $\sin(270 - A) = \sin 270 \cos A - \cos 270 \sin A$

But $\sin 270 = -1$, and $\cos 270 = 0$

Therefore $\sin(270 - A) = -\cos A$

(8) Show that $\sin(90 + A) = \cos A$

By (1), $\sin(90 + A) = \sin 90 \cos A + \cos 90 \sin A$

But $\sin 90 = 1$, and $\cos 90 = 0$

Therefore $\sin(90 + A) = \cos A$.

(9) Prove that $\sin(-A) = -\sin A$

By (3), $\sin(B - A) = \sin B \cos A - \cos B \sin A$

For B write 0

Then $\sin(-A) = \sin 0 \cos A - \cos 0 \sin A$

But $\sin 0 = 0$, and $\cos 0 = 1$

Therefore $\sin(-A) = -\sin A$

(10) Find a value for $\cos(-A)$

By (4), $\cos(B - A) = \cos B \cos A + \sin B \sin A$

For B write 0

Then $\cos(-A) = \cos 0 \cos A + \sin 0 \sin A$

or, $\cos(-A) = \cos A$

(11) To prove $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}$

By Class II — (1)

$$\text{Tan}\,(A+B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Divide numerator and denominator by $\cos A \cos B$

Then,

$$\text{Tan}\,(A+B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}} = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A \,.\, \sin B}{\cos A \,.\, \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

In like manner,

(12) $\text{Tan}\,(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(13) To show that $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$

(14) $\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$

(15) $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$

(16) $\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$

By (1), $\sin(A + B) = \sin A \cos B + \cos A \sin B$

„ (3), $\underline{\sin(A - B) = \sin A \cos B - \cos A \sin B}$

Adding, $\sin(A+B)+\sin(A-B)=2\sin A\cos B$

Subtracting, $\sin(A+B)-\sin(A-B)=2\cos A\sin B$

By (2), $\cos(A+B)=\cos A\cos B-\sin A\sin B$

„ (4), $\underline{\cos(A-B)=\cos A\cos B+\sin A\sin B}$

Adding, $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

Subtracting, $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$

(17) To show that $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

(18) $\sin A-\sin B=2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

(19) $\cos A+\cos B=2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

(20) $\cos A-\cos B=-2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$
$=2\sin\frac{1}{2}(B+A)\sin\frac{1}{2}(B-A)$

The greater, A, of any two quantities A, and B, is equal to half their sum added to half their difference: and the less, B, is equal to half their sum minus half their difference, therefore,

$$A=\left\{\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B)\right\}$$

$$\text{and } B=\left\{\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B)\right\}$$

$$\text{therefore, } \sin A=\sin\left\{\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B)\right\}$$

$$\text{and } \sin B=\sin\left\{\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B)\right\}$$

expanding the 1st, by (1)
and the 2nd, by (3)

$$\sin A=\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)+\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$\sin B=\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)-\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

Adding,

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

Subtracting,

$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

In like manner,

$$\cos A=\cos\left\{\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B)\right\}$$

$$\cos B=\cos\left\{\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B)\right\}$$

expanding the 1st, by (2)
and the 2nd, by (4)

$$\cos A=\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)-\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$\cos B=\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)+\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

Adding,

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B),$$

Subtracting,

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B)$$

Or, as it is more generally used, to avoid the negative sign,

$$\cos A - \cos B = 2 \sin \tfrac{1}{2}(B+A) \sin \tfrac{1}{2}(B-A)$$

(21) To prove, $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$

(22) $$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A+B)$$

(23) $$\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A-B)$$

By (17) and (18),

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2 \sin \frac{1}{2}(A+B) \cdot \cos \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}$$

$$= \tan \tfrac{1}{2}(A+B) \cdot \operatorname{Cot} \tfrac{1}{2}(A-B)$$

$$= \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

By (17) and (19),

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}$$

$$= \tan \tfrac{1}{2}(A+B).$$

By (18) and (19),

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}$$

$$= \tan \tfrac{1}{2}(A-B)$$

(24) To show that $\tan (45° + A) = \dfrac{1 + \tan A}{1 - \tan A}$

By (11),

$$\tan (B + A) = \frac{\tan B + \tan A}{1 - \tan B \tan A}$$

For B write 45°

$$\text{then } \tan (45° + A) = \frac{\tan 45 + \tan A}{1 - \tan 45 \cdot \tan A}$$

But by referring to the circle, we find $\tan 45° =$ radius or 1

$$\text{therefore } \tan (45 + A) = \frac{1 + \tan A}{1 - \tan A}$$

In like manner,

(25) $$\tan (45 - A) = \frac{1 - \tan A}{1 + \tan A}.$$

CLASS V.

FORMULÆ DEPENDING UPON THOSE IN THE LAST CLASS.

(1) $\text{Sin } 2A = 2 \sin A \cos A$
(2) $\text{Cos } 2A = \cos^2 A - \sin^2 A$
(3) $\text{Cos } 2A = 2 \cos^2 A - 1.$
(4) $\text{Cos } 2A = 1 - 2 \sin^2 A.$

By (1) in last class,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

For B write A, then,

$$\sin (A + A) = \sin A \cos A + \cos A \sin A$$
$$\text{or } \sin 2A = 2 \sin A \cos A$$
$$\text{and } \sin A = 2 \sin \tfrac{1}{2} A \,.\, \cos \tfrac{1}{2} A$$

By (2),

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

For B write A, then,

$$\cos (A + A) = \cos A \cos A - \sin A \,.\, \sin A$$
$$\text{or } \cos 2A = \cos^2 A - \sin^2 A$$

But by Class I, (1),

$$\cos^2 A = 1 - \sin^2 A$$
$$\therefore \cos 2A = (1 - \sin^2 A) - \sin^2 A$$
$$= 1 - 2 \sin^2 A$$

And by the same,

$$\sin^2 A = 1 - \cos^2 A$$
$$\text{therefore } \cos 2A = \cos^2 A - (1 - \cos^2 A)$$
$$= \cos^2 A - 1 + \cos^2 A$$
$$\text{or } \cos 2A = 2 \cos^2 A - 1.$$

To prove (5) $\sin 3A = 3 \sin A - 4 \sin^3 A$
(6) $\cos 3A = 4 \cos^3 A - 3 \cos A$

By (1) in last class,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

For B write 2 A

$$\text{then } \sin 3A = \sin A \cos 2A + \cos A \sin 2A$$
$$\text{But (4) } \cos 2A = (1 - 2 \sin^2 A)$$
$$\text{and (1) } \sin 2A = 2 \sin A \cos A$$
$$\text{therefore } \sin 3A = \sin A \,.\, (1 - 2 \sin^2 A) + \cos A \,.\, (2 \sin A \cos A)$$
$$= \sin A - 2 \sin^3 A + 2 \sin A \cos^2 A$$
$$= \sin A - 2 \sin^3 A + 2 \sin A \,.\, (1 - \sin^2 A)$$
$$= \sin A - 2 \sin^3 A + 2 \sin A - 2 \sin^3 A$$

or $\sin 3A = 3\sin A - 4\sin^3 A$

Again by (2) in last class,

$\text{Cos}(A + B) = \cos A \cos B - \sin A \sin B$

For B write 2 A

then $\cos 3A = \cos A \cos 2A - \sin A \sin 2A$

But (3), $\cos 2A = 2\cos^2 A - 1$

and (1), $\sin 2A = 2\sin A \cos A$

therefore $\cos 3A = \cos A(2\cos^2 A - 1) - \sin A(2\sin A \cos A)$

$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

But $\sin^2 A = 1 - \cos^2 A$

$= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

therefore $\cos 3A = 4\cos^3 A - 3\cos A$

(7) To show $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

By (11) in last class,

$$\text{Tan}(A + B) = \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}$$

For B, write A, then,

$$\text{Tan } 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

(8) To show that $1 + \cos A = 2\cos^2 \frac{1}{2}A$

(9) and $1 - \cos A = 2\sin^2 \frac{1}{2}A$

By class I, (1),

$$1 = \cos^2 A + \sin^2 A$$

and (2) $\cos 2A = \cos^2 A - \sin^2 A$

Adding, $1 + \cos 2A = 2\cos^2 A$

Subtracting, $1 - \cos 2A = 2\sin^2 A$

writing $\frac{1}{2}A$ for A, in each,

$1 + \cos A = 2\cos^2 \frac{1}{2}A$

$1 - \cos A = 2\sin^2 \frac{1}{2}A.$

From these, it is manifest that,

$$\frac{1 - \cos A}{1 + \cos A} = \frac{2\sin^2 \frac{1}{2}A}{2\cos^2 \frac{1}{2}A} = \tan^2 \tfrac{1}{2}A$$

CLASS VI.

VALUES OF CERTAIN ANGLES.

To show that,

(1) Sin 0° = 0; (2) sin 30° = $\frac{1}{2}$; (3) sin 45° = $\frac{1}{2}\sqrt{2}$

(4) Sin 60° = $\frac{1}{2}\sqrt{3}$: (5) sin 90° = 1; (6) sin 180° = 0

(1) Sin 0° = 0

Since sin (A − B) = sin A . cos B − cos A . sin B

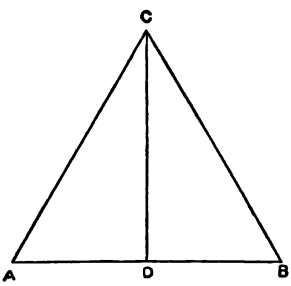
For B write A

then sin (A − A) = sin A cos A − cos A sin A

(1) Or sin 0° = 0.

(2) sin 30° = $\frac{1}{2}$

Fig. 5.



Let ABC be an equilateral triangle. Bisect the angle ACB by CD (Euc. I, 9) which will bisect AB in D (Euc. I, 4).

Then since the three angles of every triangle are together equal to two right angles (180°) (Euc. I, 32): and since the angles of an equilateral triangle are equal, therefore each angle is 60°.

Therefore ACB = 60°: or ACD which is its half, (construction) = 30°.

Again, since AB is bisected in D, therefore AD = $\frac{1}{2}$ AB = $\frac{1}{2}$ AC, and

$$\text{Sin ACD} = \frac{\text{AD}}{\text{AC}} = \frac{\text{AD}}{2\,\text{AD}} = \frac{1}{2}.$$

Another Proof.

Since sin 2A = 2 sin A . cos A

For A write 30°.

Then sin 60° = 2 sin 30° cos 30°.

But since cos A = sin (90 − A)

therefore cos 30 = sin 60°

hence sin 60° = 2 sin 30° . sin 60°

Dividing by sin 60°

$1 = 2 \sin 30°$

and $\frac{1}{2} = \sin 30°$.

(3) $\text{Sin } 45° = \frac{1}{2}\sqrt{2}$

Since $\sin {}^2A + \cos {}^2A = 1$.

For A write 45°.

Then $\sin {}^2 45° + \cos {}^2 45° = 1$.

But by referring to the circle; when the arc is 45°, its sin and cos are equal to each other,

Therefore $\sin {}^2 45° + \sin {}^2 45° = 1$

or $2 \sin {}^2 45 = 1$.

$\sin {}^2 45 = \frac{1}{2}$.

and $\sin 45° = \sqrt{\frac{1}{2}}$

But $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}.\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}$

(4) $\text{Sin } 60° = \frac{1}{2}\sqrt{3}$

Since $\sin {}^2A + \cos {}^2A = 1$

For A write 60°

Then $\sin {}^2 60° + \cos {}^2 60° = 1$.

But $\cos 60° = \sin 30° = \frac{1}{2}$

therefore $\cos {}^2 60 = \sin {}^2 30° = \frac{1}{4}$.

hence $\sin {}^2 60 + \frac{1}{4} = 1$

or $\sin {}^2 60° = \frac{3}{4}$

and $\sin 60° = \sqrt{\frac{3}{4}}$.

And $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$.

(5) $\text{Sin } 90° = 1$

Since $\sin 2A = 2 \sin A \,.\, \cos A$

For A write 45°, then,

$\text{Sin } 90° = 2 \sin 45° \,.\, \cos 45°$

But $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2}$

therefore $\sin 90 = 2 \times \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{2}$

$= 2 \times \frac{1}{4} \times 2$

or $\sin 90° = 1$.

(6) $\text{Sin } 180° = 0$.

Since $\sin A = \sin (180 - A)$

For A write 180°, then,

$\text{Sin } 180° = \sin 0° = 0$.

Having determined the values of these sines, their cosines may be deduced from this formula.

$$\text{Cos } A = \sqrt{1 - \sin^2 A} \quad \ldots \text{(Class I)}$$

Thus,

(7) $\text{Cos } 0° = \sqrt{1 - \sin^2 0} = \sqrt{1 - 0} = 1.$

(8) $\text{Cos } 30 = \sqrt{1 - \sin^2 30} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$

(9) $\text{Cos } 60° = \sqrt{1 - \sin^2 60} = \sqrt{1 - (\frac{1}{2}\sqrt{3})^2}$

$$= \sqrt{1 - \tfrac{3}{4}} = \sqrt{\tfrac{1}{4}} = \tfrac{1}{2}$$

Or otherwise,

$$\text{Cos } 60° = \sin(90° - 60°) = \sin 30° = \tfrac{1}{2}.$$

(10) $\text{Cos } 90 = \sqrt{1 - \sin^2 90} = \sqrt{1 - 1} = 0.$

But (11) $\text{Cos } 180° = \sin(90 - 180°) = \sin(-90°)$

$$= -\sin 90°$$

$$= -1.$$

The values of tangents may be thus obtained:

$$\text{Tan } A = \frac{\sin A}{\cos A}$$

i. e., (12) $\text{Tan } 0° = \dfrac{\sin 0°}{\cos 0°} = \dfrac{0}{1} = 0.$

(13) $\text{Tan } 30° = \dfrac{\sin 30°}{\cos 30°} = \dfrac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3} = \frac{1}{3}\sqrt{3}.$

(14) $\text{Tan } 45° = \dfrac{\sin 45°}{\cos 45°} = \dfrac{\sin 45°}{\sin 45°} = 1.$

(15) $\text{Tan } 60° = \dfrac{\sin 60°}{\cos 60°} = \dfrac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}.$

(16) $\text{Tan } 90° = \dfrac{\sin 90°}{\cos 90°} = \dfrac{1}{0} = \infty$ (infinity)

(17) $\text{Tan } 180° = \dfrac{\sin 180°}{\cos 180°} = \dfrac{0}{-1} = 0.$

(18) Again,

$$\text{Sec } A = \frac{1}{\cos A},$$

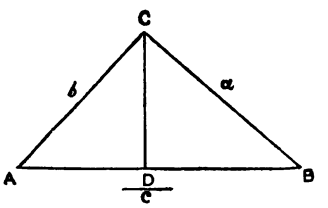
$$\text{hence sec } 60° = \frac{1}{\cos 60°} = \frac{1}{\frac{1}{2}} = 2$$

CLASS VII.

PLANE TRIANGLES.

(1) To prove that,
The sides of a plane triangle, are to each other as the sines of their opposite angles.

Fig. 6.



Let ABC be a plane triangle: from C draw CD at right angles to AB. Call the sides, opposite their respective angles, a, b, c.

Then, in the triangle ADC,

$$\text{Sin A} = \frac{\text{DC}}{\text{AC}} = \frac{\text{DC}}{b}$$

In the triangle DCB

$$\text{Sin B} = \frac{\text{DC}}{\text{BC}} = \frac{\text{DC}}{a}.$$

$$\text{Therefore } \frac{\text{Sin A}}{\text{Sin B}} = \frac{\text{DC}}{b} \times \frac{a}{\text{DC}} = \frac{a}{b}.$$

$$\text{Or Sin A} : \text{sin B} :: a : b.$$

(2) In any plane triangle, as the sum of any two sides is to their difference, so is the tangent of half the sum of the opposite angles, to the tangent of half their difference.

From the preceding, we have,

$$\frac{a}{b} = \frac{\sin \text{A}}{\sin \text{B}}.$$

Add 1 to both sides,

$$\text{then } \frac{a}{b} + 1 = \frac{\sin \text{A}}{\sin \text{B}} + 1.$$

$$\text{or } \frac{a+b}{b} = \frac{\sin \text{A} + \sin \text{B}}{\sin \text{B}} \quad \ldots (1.)$$

Subtracting 1 from both sides,

$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1.$$

$$\frac{a - b}{b} = \frac{\sin A - \sin B}{\sin B} \ldots (2.)$$

Dividing (1) by (2)

$$\frac{a + b}{b} \times \frac{b}{a - b} = \frac{\sin A + \sin B}{\sin B} \times \frac{\sin B}{\sin A - \sin B}$$

$$\text{or } \frac{a + b}{a - b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

But by Class IV. 21,

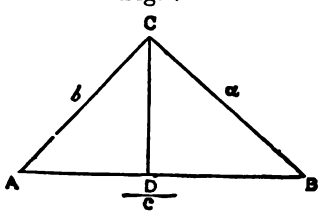
$$\frac{\text{Sin A} + \sin B}{\text{Sin A} - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$$

therefore $(a + b) : (a - b) :: \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B)$

(3) In any plane triangle, to find an expression for the cosine of one of its angles, in terms of the three sides; or in the triangle ABC, to prove,

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2bc}.$$

Fig. 7.



1st. When angle A, is acute.

Let ABC be a plane triangle, whose sides are a, b, c, respectively. From C, draw CD at right angles to AB,

Then by (Euc. II. XIII),

$$BC^2 = BA^2 + AC^2 - 2\,BA \,.\, AD$$

$$\text{or } a^2 = c^2 + b^2 - 2\,c \,.\, AD.$$

$$\text{But AD} = AC \cos A$$

$$= b \,.\, \cos A$$

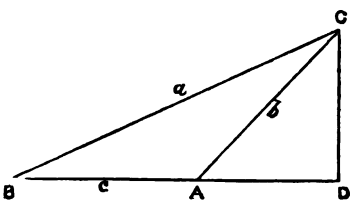
$$\text{therefore } a^2 = c^2 + b^2 - 2\,c\,b \cos A$$

$$\text{or } 2\,b\,c \cos A = b^2 + c^2 - a^2$$

$$\text{and } \cos A = \frac{b^2 + c^2 - a^2}{2\,bc.}$$

2nd. When angle A is obtuse.

Fig. 8.



By (Euc. II. XII),

$$\begin{aligned} BC^2 &= BA^2 + AC^2 + 2\,BA \,.\, AD \\ \text{or } a^2 &= c^2 + b^2 + 2c \,.\, AD \\ \text{But } AD &= AC \,.\, \cos CAD \\ &= b \,.\, \cos CAD \\ \text{and } \cos CAD &= -\cos CAB \\ &= -\cos A \\ \text{therefore } AD &= -b \,.\, \cos A \\ \text{and } a^2 &= c^2 + b^2 - 2bc\cos A \\ \text{or } 2bc\cos A &= b^2 + c^2 - a^2 \\ \text{and } \cos A &= \frac{b^2 + c^2 - a^2}{2bc}. \end{aligned}$$

(4) To find the cosine of half an angle in terms of the sides; or to prove,

$$\text{Cos } \frac{A}{2} = \sqrt{\frac{S \,.\, (S - a)}{bc}}.$$

$$\text{Since } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Add 1 to both sides, then,

$$\begin{aligned} 1 + \cos A &= 1 + \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b + c)^2 - a^2}{2bc} \\ &= \frac{(b + c + a)\,(b + c - a)}{2bc} \end{aligned}$$

Let $b + c + a = 2\,S$

Then taking $2\,a$ from both sides,

$$b + c - a = 2\,S - 2\,a = 2\,(S - a)$$

$$\text{therefore } 1 + \cos A = \frac{2\,S \,.\, 2\,(S - a)}{2\,b\,c}$$

$$\text{But } 1 + \cos A = 2\cos^2 \tfrac{1}{2} A \text{ (Class V. 8)}$$

$$\text{hence } 2\cos^2 \tfrac{1}{2} A = \frac{2\,S \,.\, 2\,(S - a)}{2\,b\,c}$$

$$\text{and } \cos \tfrac{1}{2} A = \sqrt{\frac{S \,.\, (S - a)}{b\,c}}$$

(5) To find the sine of half an angle in terms of the sides; or to prove,

$$\text{Sin } \tfrac{1}{2} A = \sqrt{\frac{(S - b)\,(S - c)}{b\,c}}$$

$$\text{Since } \cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

Subtract both sides from 1, then,

$$1 - \cos A = 1 - \left(\frac{b^2 + c^2 - a^2}{2\,b\,c}\right)$$

$$= \frac{2\,b\,c - b^2 - c^2 + a^2}{2\,b\,c}$$

$$= \frac{a^2 - b^2 - c^2 + 2\,b\,c}{2\,b\,c}$$

$$= \frac{a^2 - (b^2 + c^2 - 2\,b\,c)}{2\,b\,c}$$

$$= \frac{a^2 - (b - c)^2}{2\,b\,c}$$

$$= \frac{(a + b - c)\,(a - b + c)}{2\,b\,c}$$

As before, if $(b + c + a) = 2\,S$

then, „ $(a + b - c) = 2\,(S - c)$

and „ $(a + c - b) = 2\,(S - b)$

$$\text{therefore } 1 - \cos A = \frac{2\,(S - c)\,2\,(S - b)}{2\,b\,c}$$

$$\text{But } 1 - \cos A = 2\sin^2 \tfrac{1}{2} A \text{ (Class V. 9)}$$

$$\text{hence } 2\sin^2 \tfrac{1}{2} A = \frac{2\,(S - b) \,.\, 2\,(S - c)}{2\,b\,c}$$

$$\text{whence } \sin \tfrac{1}{2} A = \sqrt{\frac{(S - b)\,(S - c)}{b\,c}}$$

...ut, that since

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(S-b)(S-c)}{bc}}$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{S.(S-a)}{bc}}$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(S-b)(S-c)}{bc}} \div \sqrt{\frac{S.(S-a)}{bc}} = \sqrt{\frac{(S-b)(S-c)}{S.(S-a)}}$$

...f the sine of an angle in terms of the sides, or

$$\sin A = \frac{2\sqrt{S.(S-a)(S-b)(S-c)}}{bc}$$

$$\sin 2A = 2 \sin A . \cos A \text{ (Class V. 1).}$$

$$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A$$

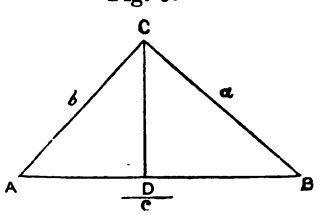
$$= 2\sqrt{\frac{(S-b)(S-c)}{bc}} \times \sqrt{\frac{S(S-a)}{bc}}$$

$$(7)\ \text{Sin } A = \frac{2\sqrt{S.(S-a)(S-b)(S-c)}}{bc}$$

... last four expressions, viz., (4), (5), (6), (7), being composed ... products," are adapted to logarithmic computation, since the ...ltiplication of quantities is performed, by the addition of their ...garithms, and division by the difference.

(8) Expressions for the area of a triangle.

Fig. 9.



Let ABC be a plane triangle, whose sides are a, b, c: it is required to find expressions for its area.

From C, draw CD at right angles to AB; then, since the area of a triangle is equal to half its base, multiplied by the perpendicular on the base from the opposite angle, therefore,

The area of ABC $= \frac{1}{2} c \times$ CD.

But CD $= b$. sin A

Hence area $= \frac{1}{2} bc$. sin A (I)

or, when two sides and their included angle are known, the area may be found, by multiplying half the product of the sides, by the sine of the angle.

* NOTE.—The area of a rectangle is found by multiplying two adjacent sides together: therefore the area of the triangle formed by drawing the diameter of the rectangle, will be half that of the rectangle, or ½ base × perpendicular.

$$\text{But } \sin A = \frac{2\sqrt{S.(S-a)(S-b)(S-c)}}{bc}$$

Substituting this, in (I), we have,

$$\text{Area} = \tfrac{1}{2}bc \times \frac{2\sqrt{S.(S-a)(S-b)(S-c)}}{bc}$$

$$\text{or, Area} = \sqrt{S.(S-a)(S-b)(S-c)}, \text{ (II)},$$

where S, is half the sum of the three sides.

CLASS VIII.

Miscellaneous Formulæ.

To show that,

$$\text{Sin}^2A - \sin^2B = \sin(A+B).\sin(A-B)$$

$$\text{Sin}^2A - \sin^2B = \sin^2A - \sin^2A.\sin^2B + \sin^2A\sin^2B - \sin^2B$$
$$= \sin^2A.(1-\sin^2B) - \sin^2B.(1-\sin^2A)$$

or,

$$\sin^2A - \sin^2B = \sin^2A\cos^2B - \cos^2A\sin^2B$$
$$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$$
$$= \sin(A+B).\sin(A-B)$$

Or, by another method,

$$\sin^2A - \sin^2B = (\sin A + \sin B).(\sin A - \sin B)$$
$$= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$= 2\sin\tfrac{1}{2}(A+{}^*B)\cos\tfrac{1}{2}(A+B).2\sin\tfrac{1}{2}({}^*A-B)\cos\tfrac{1}{2}(A-B)$$
$$= {}^*\sin(A+B).{}^*\sin(A-B)$$

* Since the sine of any angle is equal to twice the sine of its half, multiplied by the cosine of its half; . . . *eg.*, $\sin A = 2\sin\frac{1}{2}A\cos\frac{1}{2}A$.

Again, to prove,

$$\cos^2A - \cos^2B = \sin(B+A)\sin(B-A)$$
$$= \cos^2A - \cos^2A\cos^2B + \cos^2A\cos^2B - \cos^2B$$
$$= \cos^2A(1-\cos^2B) - \cos^2B(1-\cos^2A)$$
$$= \cos^2A.\sin^2B - \cos^2B\sin^2A$$

or,

$$= \sin^2B\cos^2A - \cos^2B.\sin^2A$$
$$= (\sin B\cos A + \cos B\sin A)(\sin B\cos A - \cos B\sin A)$$
$$= \sin(B+A).\sin(B-A)$$

or,

$$= -\sin(A+B).\sin(A-B)$$

SPHERICAL TRIGONOMETRY.

Definitions.

1. A plane surface is that in which any two points being taken, the line joining those points lies wholly in that surface.

2. A straight line is said to be at right angles to a plane, when it makes right angles with every line which meets it in that plane.

3. The inclination of a plane to a plane, is the acute angle contained by two straight lines drawn perpendicular to their common section from the same point in the section, one in the one plane, and the other in the other plane.

4. A solid angle is formed by the meeting in one point of more than two plane angles.

5. A sphere is a solid figure whose diameters are all equal; it may be conceived to be generated by the revolution of a semicircle about its diameter.

6. A great circle is one drawn upon the surface of a sphere dividing it into two equal parts.

7. A less circle is one drawn upon the surface of a sphere, but not dividing the sphere into two equal parts.

8. A spherical triangle is formed on the surface of a sphere, by the arcs of three great circles intersecting each other.

9. The pole of a circle is a point on the surface of a sphere on which the circle is drawn, everywhere equidistant from its circumference.

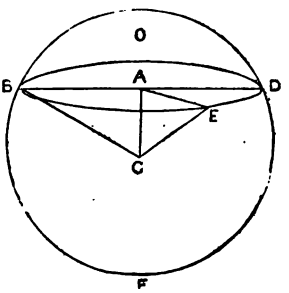
10. The centre of every great circle is the centre of the sphere on which the circle is drawn.

11. All great circles bisect one another.

Proposition I.

Every section of a sphere, made by a plane, is a circle.

Fig. 10.



Let the sphere BFD be cut through in one direction by the plane surface BODE, then the section of the sphere thus exposed will be a circle.

1st. If the section passes through the centre C of the sphere, then it is manifest that the surface forms a plane figure, every point of whose edge is equally distant from the centre C, and the figure must therefore be a circle. (Def. 5).

2nd. But if the plane does not pass through the centre of the sphere, let it pass otherwise, as BD.

Join BD. Find C the centre of the sphere. Draw CA perpendicular to BD. In the circumference BD, of the section, take a point E. Join CE and CB.

Then the line CA being at right angles to BD, it is also at right angles to AE, and to every line in that plane, drawn from A. (Def. 2). Hence CAB, and CAE are two right angled triangles, and by Euc. I, 47,

$$CE^2 = CA^2 + AE^2$$
$$\text{Also } CB^2 = CA^2 + AB^2$$

But since the diameters of a sphere are all equal (Def. 5), therefore the radii, (halves of the diameters), CE and CB are equal; and $CE^2 = CB^2$, hence, in the above,

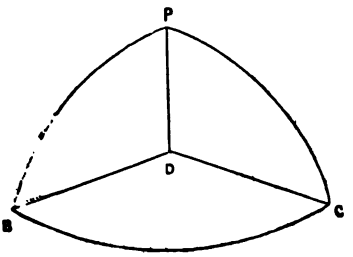
$$CA^2 + AE^2 = CA^2 + AB^2$$
$$\text{whence } AE^2 = AB^2$$
$$\text{and } AE = AB.$$

In like manner, by taking other points in the circumference of the section, it could be shown that they would be equidistant from A: hence the figure is a circle.

Proposition II.

The angle at the pole of a great circle is measured by the arc which subtends it.

Fig 11.



Let BC be an arc of a great circle, whose pole is P, and let PB and PC be the arcs of two great circles passing through P. Then the angle BPC is measured by the arc BC.

Find D the centre of the sphere on which the circles are drawn, and join DP, DB, DC.

Then since the angles BDC and BPC, both measure, the inclination of the planes BDP and PDC, therefore the angle BDC = the angle BPC.

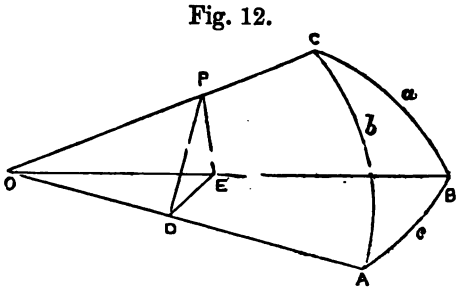
But since D is the centre of the circle BC, (Def. 10), therefore the angle at D is measured by BC (fig 1). Therefore the angle BPC is also measured by BC.

FORMULÆ.

(1) To find the cosine of an angle of a spherical triangle, in terms of the cosines of the sides, or to prove, that,

$$\text{Cos A} = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}$$

Fig. 12.



Let ABC be a spherical triangle, whose sides are a,b,c. Find O, the centre of the sphere. Join OA, OB, OC.

Then three planes are formed, viz., COA, BOA, COB. In the common section OC, take a point P. From P draw PD perpendicular to OA. From D erect DE perpendicular to OA; and join PE.

Then the solid angle at O is composed of three plane angles, which are measured by the opposite arcs, a, b, c, respectively. Then by referring to Plane Trigonometry, (3), Class VII., it will be seen that,

In the triangles POE, PDE,

$$PE^2 = PO^2 + OE^2 - 2\,PO\,.\,OE\,.\,\cos POE$$
$$PE^2 = PD^2 + DE^2 - 2\,PD\,.\,DE\,.\,\cos PDE$$

But POE is measured by a;

and PDE measures the same inclination as A; therefore,

$$PE^2 = PO^2 + OE^2 - 2\,PO\,.\,OE\,.\cos a\,.\,(1)$$
$$PE^2 = PD^2 + DE^2 - 2PD\,.\,DE\,.\cos A\,.\,(2)$$

Subtracting (2) from (1),

$$0 = PO^2 - PD^2 + OE^2 - DE^2 - 2\,PO\,.\,OE\cos a + 2\,PDDE\cos A$$

Now, since PDO is a right angle,

$$PO^2 = PD^2 + DO^2 \text{ (Euc. I, 47)}$$
$$\text{and } PO^2 - PD^2 = OD^2$$

In a similar manner, since ODE is a right angle, therefore,

$$OE^2 = OD^2 + DE^2$$
$$\text{and } OE^2 - DE^2 = OD^2$$

Substituting these, we have,

$$0 = OD^2 + OD^2 - 2\,PO\,.\,OE\cos a + 2\,PD\,.\,DE\cos A.$$
$$0 = 2\,OD^2 - 2\,PO\,.\,OE\cos a + 2\,PD\,.\,DE\cos A$$
$$= 2\,OD\,.\,OD - 2\,PO\,.\,OE\cos a + 2\,PD\,.\,DE\cos A$$

transposing,

$$2\,PO\,.OE\cos a = 2\,OD\,.\,OD + 2\,PDDE,\cos A$$

and dividing by 2 PO . OE,

$$\text{Cos } a = \frac{OD\,.\,OD}{OP\,.\,OE} + \frac{PD\,.\,DE}{PO\,.\,OE}\,.\,\cos A.$$

Remembering that ODP and ODE are right angles, and that POD and DOE are respectively measured by b, and c, we have,

$$\text{Cos } a = \cos b\,.\,\cos c + \sin b\,.\,\sin c\,.\,\cos A$$

whence,

$$\text{Cos } a - \cos b\cos c = \sin b\sin c\cos A.$$

and

$$\frac{\text{Cos } a - \cos b\cos c}{\text{Sin } b\,.\,\sin c} = \cos A.$$

(2) To find a value for the cosine of half an angle, or to prove that,

$$\text{Cos}\,\frac{A}{2} = \sqrt{\frac{\sin S \,.\, \sin (S - a)}{\sin b \,.\, \sin c}}$$

$$\text{Since } \cos A = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}$$

Add 1 to both sides then,

$$1 + \cos A = 1 + \frac{\cos a - \cos b \,.\, \cos c}{\sin b \sin c}$$

$$= \frac{\sin b \sin c + \cos a - \cos b \cos c}{\sin b \sin c}$$

Writing this in order, we have,

$$1 + \cos A = \frac{\cos a - \cos b \cos c + \sin b \sin c}{\sin b \sin c}$$

Bracketing the last terms,

$$= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \,.\, \sin c}$$

$$= \frac{2 \sin \tfrac{1}{2} (b + c + a) \,.\, \sin \tfrac{1}{2} (b + c - a)}{\sin b \sin c}$$

Let $\frac{1}{2} (b + c + a) = S$

then $\frac{1}{2} (b + c - a = (S - a)$

and $1 + \cos A = 2 \cos^2 \frac{1}{2} A$

$$\text{hence, } 2 \cos^2 \tfrac{1}{2} A = \frac{2 \sin S \,.\, \text{Sin}\,(S - a)}{\sin b \sin c}$$

$$\text{and } \cos \tfrac{1}{2} A = \sqrt{\frac{\sin S \,.\, \sin (S - a)}{\sin b \,.\, \sin c}}$$

(3). To find the sine of half an angle, in terms of the sides, or to prove, that,

$$\text{Sin}\,\tfrac{1}{2} A = \sqrt{\frac{\sin (S - b) \sin (S - c)}{\sin b \sin c}}$$

Again, since

$$\text{Cos}\,A = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \sin c}$$

Subtract both sides from 1,

$$\text{then } 1 - \cos A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)$$

$$= \frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}$$

$$= \frac{2 \sin \tfrac{1}{2}(a + b - c) \sin \tfrac{1}{2}(a + c - b)}{\sin b \sin c}$$

Then if $\frac{1}{2}(b + c + a) = S$,

$\frac{1}{2}(a + b - c) = (S - c)$

and $\frac{1}{2}(a + c - b) = (S - b)$

and $1 - \cos A = 2 \sin^2 \frac{1}{2} A$

therefore,

$$2 \sin^2 \tfrac{1}{2} A = \frac{2 \sin (S - b) \sin (S - c)}{\sin b \sin c}$$

$$\text{or } \sin \tfrac{1}{2} A = \sqrt{\frac{\sin (S - b) \sin (S - c)}{\sin b \,.\, \sin c}}$$

(4) To prove $\tan \frac{1}{2} A = \sqrt{\frac{\sin (S - b) (\sin S - c)}{\sin S \,.\, \sin (S - a)}}$

Since

$$\text{Tan } \tfrac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A}.$$

and,

$$\text{Sin } \tfrac{1}{2} A = \sqrt{\frac{\sin (S-b) \sin (S-c)}{\sin b \sin c}}$$

also,

$$\text{Cos } \tfrac{1}{2} A = \sqrt{\frac{\sin S \,.\, \sin (S-a)}{\sin b \,.\, \sin c}}$$

therefore,

$$\text{Tan } \tfrac{1}{2} A = \sqrt{\frac{\frac{\sin (S-b) \sin (S-c)}{\sin b \,.\, \sin c}}{\frac{\sin S \,.\, \sin (S-a)}{\sin b \,.\, \sin c}}} = \sqrt{\frac{\sin (S-b) \sin (S-c)}{\sin S \,.\, \sin (S-a)}}$$

(5) To prove,

$$\text{Sin } A = \frac{2 \sqrt{\sin S \,.\, \sin (S - a) \sin (S - b) \sin (S - c)}}{\sin b \,.\, \sin c}$$

Since

$$\text{Sin A} = 2 \sin \tfrac{1}{2} \text{A} \,.\, \cos \tfrac{1}{2} \text{A}$$

$$= 2 \sqrt{\frac{\sin (\text{S}-b) \,.\, \sin (\text{S}-c)}{\sin b \,.\, \sin c}} \times \sqrt{\frac{\sin \text{S} \,.\, \sin (\text{S}-a)}{\sin b \sin c}}$$

therefore,

$$\text{Sin A} = 2 \frac{\sqrt{\sin \text{S} \,.\, \sin (\text{S}-a) \sin (\text{S}-b) \sin (\text{S}-c)}}{\sin b \,.\, \sin c}$$

(6) In any spherical triangle the sines of the sides are to one another, as the sines of their opposite angles; or

$$\text{Sin } a : \text{sin } b :: \text{sin A} : \text{sin B}$$

By last formula,

$$\text{Sin A} = \frac{2\sqrt{\sin \text{S} \,.\, \sin (\text{S}-a) \sin (\text{S}-b) \,.\, \sin (\text{S}-c)}}{\sin b \,.\, \sin c}$$

In like manner,

$$\text{Sin B} = \frac{2\sqrt{\sin \text{S} \,.\, \sin (\text{S}-b) \sin (\text{S}-a) \sin (\text{S}-c)}}{\sin a \sin c}$$

Let the two numerators (which are the same) be called N, then,

$$\text{Sin A} = \frac{\text{N}}{\sin b \sin c} \text{ and sin B} = \frac{\text{N}}{\sin a \sin c}$$

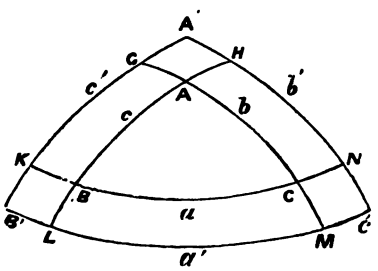
therefore,

$$\frac{\text{Sin A}}{\text{Sin B}} = \frac{\text{N}}{\sin b \sin c} \times \frac{\sin a \sin c}{\text{N}} = \frac{\sin a}{\sin b}$$

or, $\sin a : \sin b :: \text{sin A} : \text{sin B}$

(7) THE POLAR TRIANGLE.

Fig. 13.



Let ABC be any spherical triangle. Take A as a centre, and with a radius of 90°, describe the arc B′ C′. In like manner, with B, and C as centres and distances of 90° describe the arcs A′ C′, and A′ B′. Then the triangle A′ B′ C′ is called the "polar" or "supplemental" triangle.

It is called the "polar" triangle, because, as we shall prove, its three angular points are the poles of the sides opposite in the inner triangle.

It is called the "supplemental" triangle because each angle is the supplement of the opposite side of the other triangle, and its sides are supplements of the angles of the other triangle.

We have to prove three things,

1st.—The angular points of the polar triangle are the poles of the sides of the other triangle.

2nd.—The sides of the polar triangle are the supplements of the angles of the other.

3rd.—The angles of the polar triangle are the supplements of the sides of the other.

1st.—To prove A′ the pole of BC.

Because B is the pole of A′ C′ (construct^{n})
Therefore BH is a quadrant
Because C is the pole of A′ B′
Therefore CG is a quadrant.

Therefore from B and C two arcs each measuring a quadrant are drawn towards A′, therefore A′ is the pole of BC.

2nd.—To prove $B'C' = (180 - A)$

$$\begin{aligned} B'C' &= B'M + MC' \\ &= B'M + C'L - LM \\ &= 90 + 90 - A \end{aligned}$$

For LM measures A . . . (Prop. II)

Therefore $B'C' = (180 - A)$

3rd. To prove that $BC = (180 - A')$

$$\begin{aligned} KN &= KC + CN \\ &= KC + NB - BC \\ &= 90 + 90 - BC \end{aligned}$$

But KN measures A′ (Prop. II)

therefore $A' = 180 - BC$.

(8) *Napier's Analogies.*

1st Analogy.

$$\text{Tan } \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}.(a + b)} . \cot \frac{C}{2}.$$

$$\text{Since } \cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

Therefore,

$$\sin a \sin b \cos c = \cos c - \cos a \cos b .$$

$$\text{and } \cos c = \cos b \cos b + \sin a \sin b \cos C$$

$$\text{Again, } \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$\text{but } \cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$\text{therefore } \cos A = \frac{\cos a - \cos b \,.\, (\cos a \cos b + \sin a \sin b \cos C)}{\sin b \,.\, \sin c}$$

$$= \frac{\cos a - \cos a \cos^2 b - \sin a \sin b \cos b \,.\, \cos C}{\sin b \,.\, \sin c}$$

$$= \frac{\cos a (1 - \cos^2 b) - \sin a \sin b \cos b \,.\, \cos C}{\sin b \sin c}$$

$$= \frac{\cos a \,.\, \sin^2 b - \sin a \sin b \cos b \,.\, \cos C}{\sin b \sin c}$$

Divide numr and denr by $\sin b$, then,

$$\cos A = \frac{\cos a \sin b - \sin a \cos b \,.\, \cos C}{\sin c} \quad (\alpha)$$

For A write B, and for B write A, then,

$$\cos B = \frac{\cos b \,.\, \sin a - \sin b \cos a \cos C}{\sin c} \quad (\beta)$$

Adding (α) and (β)

$$\cos A + \cos B = \frac{\sin (a+b) - \sin (a+b) \,.\, \cos C}{\sin c}$$

$$= \frac{\sin (a+b) \,.\, \left\{1 - \cos C\right\}}{\sin c}$$

$$\cos A + \cos B = \frac{\sin (a+b) \,.\, 2 \sin^2 \tfrac{1}{2} C}{\sin c} \quad .. \quad (1)$$

But since the sines of the sides are to each other as the sines of their opposite angles, therefore,

$$\frac{\text{Sin A}}{\text{Sin C}} = \frac{\sin a}{\sin c}, \text{ whence } \sin A = \frac{\sin a \,.\, \sin C}{\sin c}$$

and in like manner, $\sin B = \frac{\sin b \,.\, \sin C}{\sin c}$

Adding these last two equations together,

$$\sin A + \sin B = \frac{(\sin a + \sin b) \,.\, \sin C}{\sin c.} \quad (2)$$

Dividing (2) by (1),

$$\frac{\text{Sin A} + \sin B}{\text{Cos A} + \text{Cos B}} = \frac{(\sin a + \sin b) \,.\, \sin C}{\sin (a+b) \,.\, 2 \sin^2 \tfrac{1}{2} C}$$

Expanding these terms,

$$\frac{2 \sin \tfrac{1}{2} (A+B) \cos \tfrac{1}{2} (A-B)}{2 \cos \tfrac{1}{2} (A+B) \cos \tfrac{1}{2} (A-B)} = \frac{2 \sin \tfrac{1}{2} (a+b) \cos \tfrac{1}{2} (a-b)}{\sin (a+b)} \cdot \frac{\sin C}{2 \sin^2 \tfrac{1}{2} C}$$

C

But by class V. (1), we have

$$\sin a = 2 \sin \tfrac{1}{2} a \,.\, \cos \tfrac{1}{2} a$$

$$\text{hence } \sin (a+b) = 2 \sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a+b)$$

$$\text{and } \sin C = 2 \sin \tfrac{1}{2} C \,.\, \cos \tfrac{1}{2} C$$

Therefore,

$$\frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} = \frac{2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a+b)} \cdot \frac{2 \sin \frac{1}{2} C \cdot \cos \frac{1}{2} C}{2 \sin^2 \frac{1}{2} C}$$

$$\text{or } \tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cdot \cot \frac{C}{2}$$

2nd Analogy.

Referring to 1st Analogy, we have,

$$\text{Cos A} + \cos B = \frac{\sin (a+b) \,.\, 2 \sin^2 \frac{1}{2} C}{\sin c} \quad (1)$$

and also,

$$\text{Sin A} = \frac{\sin a \sin C}{\sin c}, \text{ and } \sin B = \frac{\sin b \,.\, \sin C}{\sin c}$$

whence,

$$\text{Sin A} - \sin B = \frac{(\sin a \,. - \sin b) \,.\, \sin C}{\sin c} \quad (3)$$

Dividing (3) by (1)

$$\frac{\text{Sin A} - \sin B}{\text{Cos A} + \cos B} = \frac{(\sin a - \sin b) \,.\, \sin C}{\sin (a+b) \,.\, 2 \sin^2 \frac{1}{2} C}$$

Expanding,

$$\frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} = \frac{2 \cos \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a+b)} \cdot \frac{2 \sin \frac{1}{2} C \cos \frac{1}{2} C}{2 \sin^2 \frac{1}{2} C}$$

$$\text{therefore } \tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cdot \cot \frac{C}{2}$$

These two Analogies are used, in solving that case of spherical triangles, where two sides and the included angle are given, to find the remaining angles of the triangle. For logarithmic computation, they are thus written,

(1) $\text{Tan } \frac{1}{2}(A+B) = \cos \frac{1}{2}(a-b) \sec \frac{1}{2}(a+b) \cot \frac{1}{2} C$

(2) $\text{Tan } \frac{1}{2}(A-B) = \sin \frac{1}{2}(a-b) \operatorname{cosec} \frac{1}{2}(a+b) \cot \frac{1}{2} C$

3rd Analogy.

$$\text{Tan } \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \cdot \tan \frac{c}{2}.$$

We know, from the 1st Analogy, that,

$$\text{Tan } \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cdot \cot \frac{C}{2}$$

And as a formulæ which is true for every triangle, must be true for the "polar triangle," by accenting the letters, we have,

$$\text{Tan}\tfrac{1}{2}(A'+B')=\frac{\cos\frac{1}{2}(a'-b')}{\cos\frac{1}{2}(a'+b')}\cdot\cot\frac{C'}{2}$$

But the angles of the polar triangle are supplements of the sides of the primitive one: and its sides are supplements of the angles, therefore,

$$\tfrac{1}{2}(A'+B')=\tfrac{1}{2}\left\{\overline{180-a}+\overline{180-b}\right\}=\tfrac{1}{2}\left\{360-(a+b)\right\}$$
$$=\left\{180-\tfrac{1}{2}(a+b)\right\}$$

and,

$$\tfrac{1}{2}(a'-b')=\tfrac{1}{2}\left\{\overline{180-A}-\overline{180-B}\right\}=\tfrac{1}{2}(-A+B)$$
$$=\tfrac{1}{2}(B-A)$$

also,

$$\tfrac{1}{2}(a'+b')=\tfrac{1}{2}\left\{\overline{180-A}+\overline{180-B}\right\}=\tfrac{1}{2}\left\{360-\overline{A+B}\right\}$$
$$=\left\{180-\tfrac{1}{2}(A+B)\right\}$$

$$\text{and }\frac{C'}{2}=\frac{180-c}{2}=\left(90-\frac{c}{2}\right)$$

Substituting these values,

$$\text{Tan}\left\{180-\tfrac{1}{2}(a+b)\right\}=\frac{\cos\frac{1}{2}(B-A)}{\cos\left\{180-\frac{1}{2}(A+B)\right\}}\cot\left(90-\frac{c}{2}\right)$$

$$\text{But }\tan\left\{180-\tfrac{1}{2}(a+b)\right\}=-\tan\tfrac{1}{2}(a+b)$$

$$\text{and }\cos\tfrac{1}{2}(B-A)=\cos\tfrac{1}{2}(A-B)$$

for their expansions are equal,

$$\text{and }\cos\left\{180-\tfrac{1}{2}(A+B)\right\}=-\cos\tfrac{1}{2}(A+B)$$

$$\text{and }\cot\left(90-\frac{c}{2}\right)=\tan\frac{c}{2}$$

Therefore,

$$-\text{Tan}\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{-\cos\frac{1}{2}(A+B)}\cdot\tan\frac{c}{2}$$

Dividing both sides by -1,

$$\text{Tan}\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\cdot\tan\frac{c}{2}$$

4th Analogy.

By the 2nd Analogy,

$$\text{Tan}\tfrac{1}{2}(\text{A}-\text{B}) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cdot\cot\frac{\text{C}}{2}$$

Referring as before to the "polar triangle,"

$$\text{Tan}\tfrac{1}{2}(\text{A}'-\text{B}') = \frac{\sin\frac{1}{2}(a'-b')}{\sin\frac{1}{2}(a'+b')}\cdot\cot\frac{\text{C}'}{2}$$

Reducing by the "polar triangle."

$$\tfrac{1}{2}(\text{A}'-\text{B}') = \tfrac{1}{2}\left\{\overline{180-a}-\overline{180-b}\right\} = \tfrac{1}{2}(b-a)$$

$$\tfrac{1}{2}(a'-b') = \tfrac{1}{2}\left\{\overline{180-\text{A}}-\overline{180-\text{B}}\right\} = \tfrac{1}{2}(\text{B}-\text{A})$$

$$\tfrac{1}{2}(a'+b') = \tfrac{1}{2}\left\{\overline{180-\text{A}}+\overline{180-\text{B}}\right\} = \tfrac{1}{2}\left\{360-\overline{\text{A}+\text{B}}\right\}$$

$$= \left\{180-\tfrac{1}{2}(\text{A}+\text{B})\right\}$$

and, as before, $\frac{\text{C}'}{2} = \left(90-\frac{c}{2}\right)$

Substituting these values,

$$\text{Tan}\tfrac{1}{2}(b-a) = \frac{\sin\frac{1}{2}(\text{B}-\text{A})}{\sin\left\{180-\frac{1}{2}(\text{A}+\text{B})\right\}}\cdot\cot\left(90-\frac{c}{2}\right)$$

But $\tan\frac{1}{2}(b-a) = -\tan\frac{1}{2}(a-b)$

and $\sin\frac{1}{2}(\text{B}-\text{A}) = -\sin\frac{1}{2}(\text{A}-\text{B})$

and $\sin\left\{180-\frac{1}{2}(\text{A}+\text{B})\right\} = \sin\frac{1}{2}(\text{A}+\text{B})$

as may be found by comparing their expansions, therefore,

$$-\text{Tan}\tfrac{1}{2}(a-b) = \frac{-\sin\frac{1}{2}(\text{A}-\text{B})}{\sin\frac{1}{2}(\text{A}+\text{B})}\cdot\tan\frac{c}{2}$$

Multiplying both sides by -1.

$$\text{Tan}\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(\text{A}-\text{B})}{\sin\frac{1}{2}(\text{A}+\text{B})}\cdot\tan\frac{c}{2}$$

These two latter analogies are of use where two angles and the adjacent side of a spherical triangle are given to find the other two sides. Adapted to logarithmic computation, they are written,

(3) $\text{Tan}\frac{1}{2}(a+b) = \cos\frac{1}{2}(\text{A}-\text{B})\sec\frac{1}{2}(\text{A}+\text{B})\cdot\tan\frac{1}{2}c$

(4) $\text{Tan}\frac{1}{2}(a-b) = \sin\frac{1}{2}(\text{A}-\text{B})\,\text{cosec}\frac{1}{2}(\text{A}+\text{B})\cdot\tan\frac{1}{2}c$

(9)

To find the cosine of a side in terms of the three angles of a triangle,

$$\text{Since } \cos A = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}$$

Referring to the polar triangle,

$$\text{Cos } A' = \frac{\cos a' - \cos b' \,.\, \cos c'}{\sin b' \sin c'}$$

$$\text{But } A' = (180 - a);\ a' = (180 - A)$$

$$\text{„ } b' = (180 - B);\ c' = (180 - C)$$

Therefore,

$$\text{Cos } (180 - a) = \frac{\cos (180 - A) - \cos (180 - B) \cos (180 - C)}{\sin (180 - B) \sin (180 - C)}$$

$$\text{But } \cos (180 - a) = -\cos a$$

$$\text{and } \sin (180 - B) = \sin B$$

$$-\text{Cos } a = \frac{-\cos A - (-\cos B) \,.\, (-\cos C)}{\sin B \,.\, \sin C}$$

$$\text{whence } \cos a = \frac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C}$$

(10)

A formula, for finding the third side of a spherical triangle, when two sides and the included angle are given.

$$\text{Since } \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$\sin b \,.\, \sin c \,.\, \cos A = \cos a - \cos b \,.\, \cos c$$

$$\text{and } \cos a = \cos b \,.\, \cos c + \sin b \sin c \cos A$$

$$\text{But } \cos A = 2 \cos^2 \tfrac{1}{2} A - 1. \quad \text{(Class V. 3)}$$

$$\text{therefore } \cos a = \cos b \,.\, \cos c + \sin b \,.\, \sin c \,.\, (2 \cos^2 \tfrac{1}{2} A - 1)$$

$$= (\cos b \,.\, \cos c - \sin b \sin c) + 2 \sin b \sin c \cos^2 \tfrac{1}{2} A$$

$$\cos a = \cos (b + c) + 2 \sin b \,.\, \sin c \cos^2 \tfrac{1}{2} A$$

Subtracting both sides from 1,

$$1 - \cos a = 1 - \cos (b + c) - 2 \sin b \,.\, \sin c \cos^2 \tfrac{1}{2} A$$

$$\text{or } 2 \sin^2 \frac{a}{2} = 2 \sin^2 \frac{b + c}{2} - 2 \sin b \,.\, \sin c \,.\, \cos^2 \tfrac{1}{2} A$$

Let $\sin b \,.\, \sin c \,.\, \cos^2 \tfrac{1}{2} A = \sin^2 \theta$

$$\text{then } \sin^2 \frac{a}{2} = \sin^2 \frac{b + c}{2} - \sin^2 \theta$$

$$\text{or } \sin^2 \frac{a}{2} = \sin \left(\frac{b + c}{2} + \theta \right) \,.\, \sin \left(\frac{b + c}{2} - \theta \right) \text{ (Class VIII)}$$

$$\text{and } \sin\frac{a}{2} = \sqrt{\sin\left(\frac{b+c}{2}+\theta\right).\sin\left(\frac{b+c}{2}-\theta\right)}$$

whence, with b, c, and A given, a may be found.

(11)

A formula, for connecting four successive parts of a triangle, or to prove,

$$\text{Cot A} \,.\, \sin \text{B} = \cot a \,.\, \sin c - \cos \text{B} \,.\, \cos c$$

$$\text{Since } \cos \text{A} = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}$$

$$\text{Sin } b \,.\, \sin c \,.\, \cos \text{A} = \cos a - \cos b \,.\, \cos c$$

But

$$\frac{\text{Sin } b}{\text{Sin } a} = \frac{\sin \text{B}}{\sin \text{A}}, \text{ whence } \sin b = \frac{\sin \text{B} \,.\, \sin a}{\sin \text{A}}.$$

and $\cos b = \cos a \cos c + \sin a \sin c \cos \text{B}$

Substituting these values, we have,

$$\frac{\text{Sin B} \,.\, \sin a}{\text{Sin A}} \,.\, \sin c \,.\, \cos \text{A} = \cos a - \cos c(\cos a \cos c + \sin a \sin c \cos \text{B})$$

$$= \cos a - \cos a \cos^2 c - \sin a \sin c \cos c \cos \text{B}$$

$$= \cos a(1 - \cos^2 c) - \sin a \sin c \cos c \cos \text{B}$$

$$= \cos a \sin^2 c - \sin a \sin c \cos c \cos \text{B}$$

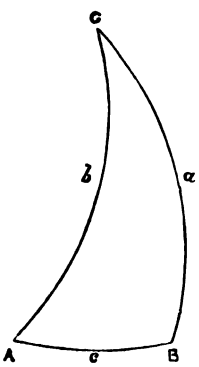
Dividing both sides by $\sin a \,.\, \sin c$, and noticing, that, on the left hand side of the equation, $\frac{\text{Cos A}}{\text{Sin A}} = \cot \text{A}$, we have,

$$\text{Cot A sin B} = \cot a \,.\, \sin c - \cos \text{B} \,.\, \cos c$$

(12)

Napier's rules for the solution of right angled spherical triangles.

Fig. 14.



Every triangle consists, strictly, of seven parts, viz., the three sides, three angles, and the area, or space enclosed by the sides.

Now, if in the right angled spherical triangle ABC, we agree to omit, the area and right angle B, we shall have five of the parts left, viz., the three sides, and angles A and C.

Taking one of these five parts as what is termed a "middle part," we have two of the remaining parts nearest it, and two most remote. The nearest are called "adjacents," and the remotest "opposites," or "extremes."

Thus, taking

b as a middle part;
a and c are extremes; A and C adjacents.
A as a middle part;
a and C are extremes; b and c adjacents.
c as a middle part;
b and C are extremes; A and a adjacents.

And so on for the rest.

From the formulæ hereafter proved, we gather two comprehensive rules, viz., that,

sin mid. part = prod. tangents of "adjacents,"
sin mid. part = prod. cosines of "extremes,"

under the supposition that when we apply these rules to b, and A and C, we use their complements $(90 - b)$, $(90 - A)$, and $(90 - C)$; thus, following out the rules:—

$$\sin (90 - b) = \cos a \,.\, \cos c\,.$$
$$\sin (90 - b) = \tan (90 - A) \tan (90 - C)$$
or (1) $\cos b = \cos a \cos c$
(2) $\cos b = \cot A \,.\, \cot C$
and $\sin (90 - A) = \cos a \,.\, \cos (90 - C)$
„ $\sin (90 - A) = \tan (90 - b) \,.\, \tan c\,.$
or (3) $\cos A = \cos a \,.\, \sin C$
(4) $\cos A = \cot b \,.\, \tan c$
and $\sin c = \cos (90 - b) \,.\, \cos (90 - C)$
„ $\sin c = \tan (90 - A) \,.\, \tan a$
or (5) $\sin c = \sin b \,.\, \sin C$
(6) $\sin c = \cot A \,.\, \tan a$ &c.

To prove (1) and (2)

Since,

$$\text{Cos B} = \frac{\cos b - \cos a \cos c}{\sin a \sin c}$$

and $B = 90°$: $\cos B = 0$.

Therefore,

$$0 = \frac{\cos b - \cos a \cos c}{\sin a \sin c}$$

or $\cos b - \cos a \cos c = 0$
and (1) $\cos b = \cos a \cos c$.

Also, since,

$$\text{Cos } b = \frac{\cos B + \cos A \cos C}{\sin A \,.\, \sin C}$$

and $\cos B = 0$

Therefore

$$(2)\ \text{Cos } b = \frac{\cos A \,.\, \cos C}{\sin A \sin C} = \cot A \,.\, \cot C$$

To prove (3) and (4)

$$\text{Since, } \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

$B = 90°$: $\cos B = 0$: $\sin B = 1$

$$\text{hence, } \cos a = \frac{\cos A}{\sin C}$$

therefore (3) $\cos a \sin C = \cos A$.

Again

$$\text{Cot B} \,.\, \sin A = \cot b \,.\, \sin c - \cos A \,.\, \cos c \ (11).$$

$$\text{Cot B} = 0\,.$$

$$0 = \cot b \,.\, \sin c - \cos A \cos c$$

$$\text{Cos A} \cos c = \cot b \,.\, \sin c$$

and (4) $\cos A = \cot b \,.\, \tan c$

To prove (5) and (6)

$$\text{Since } \frac{\sin b}{\sin c} = \frac{\sin B}{\sin C}$$

and $\sin B = 1$

$$\text{therefore } \frac{\sin b}{\sin c} = \frac{1}{\sin C}$$

and (5) $\sin b \,.\, \sin C = \sin c$

Also,

$$\text{Cot A} \,.\, \sin B = \cot a \sin c - \cos B \,.\, \cos c \ (11).$$

$\sin B = 1$: $\cos B = 0$.

therefore, $\cot A = \cot a \sin c$

$$\text{and (6) } \sin c = \frac{\cot A}{\cot a} = \cot A \,.\, \tan a$$

(13)

An expression for the cosine of half a side when the three angles are known.

$$\text{Since, } \cos a = \frac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C}$$

Adding 1 to both sides,

$$1 + \cos a = 1 + \frac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C}$$

$$= \frac{\sin B \,.\, \sin C + \cos A + \cos B \cos C}{\sin B \sin C}$$

$$= \frac{\cos B \cos C + \sin B \,.\, \sin C + \cos A}{\sin B \,.\, \sin C}$$

$$= \frac{\cos (B - C) + \cos A}{\sin B \,.\, \sin C}$$

$$\text{or } 2 \cos^2 \frac{a}{2} = \frac{2 \cos \tfrac{1}{2} (A + B - C) \,.\, \cos \tfrac{1}{2} (A - B + C)}{\sin B \,.\, \sin C}$$

Let $\frac{1}{2} (B + C + A) = M$

then $\frac{1}{2} (A + C - B) = (M - B)$

and $\frac{1}{2} (A + B - C) = (M - C)$.

$$\text{and, } \cos^2 \frac{a}{2} = \frac{\cos (M - B) \cos (M - C)}{\sin B \,.\, \sin C}$$

$$\text{whence, } \cos \frac{a}{2} = \sqrt{\cos (M-B) \cos (M-C) \operatorname{cosec} B \,.\, \operatorname{cosec} C}$$

(14)

To find an expression for the sine of half a side, when the three angles are known.

$$\text{Again, } \cos a = \frac{\cos A + \cos B \,.\, \cos C}{\sin B \sin C}$$

Subtracting both sides from 1,

$$1 - \cos a = 1 - \left(\frac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C} \right)$$

$$= \frac{\sin B \,.\, \sin C - \cos A - \cos B \cos C}{\sin B \,.\, \sin C}$$

$$= \frac{-\cos A - \cos B \,.\, \cos C + \sin B \,.\, \sin C}{\sin B \,.\, \sin C}$$

$$= \frac{-\cos A - (\cos B \cos C - \sin B \sin C)}{\sin B \sin C}$$

$$= \frac{-\cos A - \cos (B + C)}{\sin B \,.\, \sin C}$$

$$= \frac{-\left\{ \cos A + \cos (B + C) \right\}}{\sin B \,.\, \sin C}$$

$$2 \sin^2 \frac{a}{2} = - \frac{2 \cos \tfrac{1}{2} (B + C + A) \cos \tfrac{1}{2} (B + C - A)}{\sin B \,.\, \sin C}$$

Then if as before, $\frac{1}{2}(B + C + A) = M$

$$\text{Sin}\,\frac{a}{2} = \sqrt{-\cos M \cdot \cos (M - A) \cdot \text{cosec}\, B \cdot \text{cosec}\, C}$$

The former of these is the expression used in logarithmic computation.

PROPOSITIONS *(continued)*.

Proposition II

(Otherwise proved).

The angle at the pole of a great circle is measured by the arc which subtends it. (See fig. 11).

$$\text{Since } \cos P = \frac{\cos BC - \cos BP \cos PC}{\sin BP \cdot \sin PC}$$

and BP and PC are quadrants,

$$\text{therefore } \cos P = \frac{\cos BC - \cos 90^\circ \cos 90^\circ}{\sin 90^\circ \sin 90^\circ}$$

But $\cos 90^\circ = 0$, and $\sin 90^\circ = 1$

hence $\cos P = \cos BC$,

or $P = BC$

Proposition III.

The angles of a spherical triangle are together greater than two and less than six right angles.

Referring to the figure of the polar triangle, we have,

$$A = 180 - a'$$
$$B = 180 - b'$$
$$\text{and } C = 180 - c'.$$

$$\text{therefore, } A + B + C = 540^\circ - (a' + b' + c')$$

And taking into consideration the truth, that the sum of the three sides of a spherical triangle is less than the circumference of a great circle (360°), therefore,

$A + B + C = 540^\circ$ — a quantity less than 360°

But $540^\circ - 360^\circ = 180^\circ$

Therefore, since this quantity is less than 360°

$A + B + C$ must be greater than 180° or two right angles

And it is evidently less than six for $(a' + b' + c')$ is taken from 540° to equal $(A + B + C)$.

Proposition IV.

The angles at the base of an isosceles spherical triangle are equal.

Let ABC be any isosceles spherical triangle, and let $a = b$, then $A = B$.

$$\text{Since } \frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$$

and $a = b$; therefore $\sin a = \sin b$

whence $\sin A = \sin B$

and $A = B$

Proposition V.

The greater side of every spherical triangle is opposite the greater angle.

In any spherical triangle ABC let a be greater than b, then A is greater than B

By the same formula,

$$\frac{\text{Sin } a}{\text{Sin } b} = \frac{\sin A}{\sin B}$$

But a is greater than b

therefore $\sin a$ is greater than $\sin b$

or $\sin A$ is greater than $\sin B$

and A is greater than B.

INVESTIGATIONS OF RULES

IN

NAVIGATION

AND

NAUTICAL ASTRONOMY.

INVESTIGATIONS OF RULES

IN

NAVIGATION

AND

NAUTICAL ASTRONOMY.

Definitions in Navigation.

1. The extremities of the Earth's axis are the **Poles.**

2. The **Equator** is a great circle passing round the Earth, at an equal distance from each pole.

3. **Meridians** are great circles passing through both poles, perpendicular to the equator.

4. **Parallels of latitude** are small circles parallel to the equator.

5. The **Tropic of Cancer** is that parallel of latitude $23\frac{1}{2}°$ north of the equator.

6. The **Tropic of Capricorn** is that $23\frac{1}{2}°$ south.

7. **Latitude** is an arc of a meridian intercepted between the place and the equator. (Never exceeds 90°).

8. **Difference of latitude** between two places is the arc of a meridian intercepted between the parallels of latitude drawn through the places.

9. **Longitude** is an arc of the equator intercepted between the "first meridian" and the meridian, over the place. (Never exceeds 183°).

10. **Difference of longitude** is the arc of the equator intercepted between the meridians, over the two places. (Never exceeds 180°).

11. A **rhumb line** is a line cutting all meridians at the same angle.

12. The **Course** of a ship is the angle which her track through the water makes with the meridian.

13. The **Meridian distance** between two places is the arc of a parallel of latitude between them.

14. **Departure** is the sum of all the intermediate meridian distances made in going from one place to another.

15. The **Steering** or **compass course** is obtained from the true course by applying the variation and deviation of the compass.

16. **Leeway** is the angle between a ship's actual course and her intended course, the change of direction being caused by winds or currents.

17. A ship is said to be "on the **port tack**" when the wind is on her port side; and "on the **starboard** tack" when it is blowing upon the starboard side.

18. The opposite to the "windward side" is the "**lee side.**"

19. A ship is **close hauled** when sailing as near the wind as she will lie.

NAUTICAL ASTRONOMY.

(Theory).

The Earth on which we live is nearly a sphere (an oblate spheroid), the difference between its major and minor diameters being 26 miles.

It is the third planet of the solar system, and accomplishes a revolution round the Sun, in about 365¼ days. The path in which it moves is elliptical in form, and is called its "orbit," and the Sun occupies one of the "foci" of this ellipse.

Besides its motion round the Sun, it has one round its axis or shortest diameter, from west to east, which is accomplished in 24 hours. By the former motion we have the phenomena of the seasons; and by the latter, those of day and night.

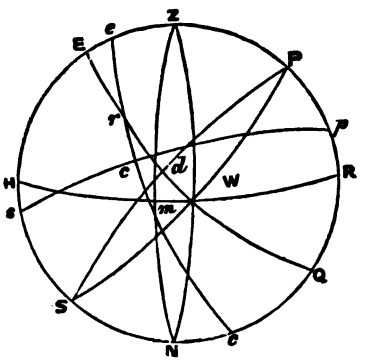
Now, if the **Earth** be supposed the stationary object, and not the Sun, the latter body will appear to perform the Earth's real motion, but in a contrary direction.

The Sun is said to move round the Earth performing both a daily and yearly course, rising in the easterly part of the heavens, and setting in the westerly quarter; and also moving

during the twelve months, in a complete circle amongst certain groups of stars, called the "Constellations of the Zodiac."

Definitions in Nautical Astronomy.

Fig. 15.



Explanation of Figure.

Celestial Meridian, HZRN.	Circle of Altitude, ZN.
Horizon, HR.	Circle of Declⁿ. *Pds.*
Equator, EQ.	Circle of Latitude, *pcs.*
Prime Vertical, ZWN.	1st pt. of Aries, *r.*
Six o'clock hr. Circle, PWS.	Poles of Ecliptic, *p. s.*
Ecliptic, *ec.*	Poles of Heavens, P. S.
Zenith, Z.	Nadir, N.

(1) *Circles.*

1. The **celestial meridian** is the meridian of an observer on the Earth, produced to meet the celestial concave.

2. The **sensible horizon** is the plane on which the observer stands, produced to meet the celestial concave.

3. The **rational horizon** is an imaginary plane parallel to the sensible horizon, and passing through the centre of the Earth.

4. The **celestial equator** is the equator of the Earth, expanded to meet the celestial concave.

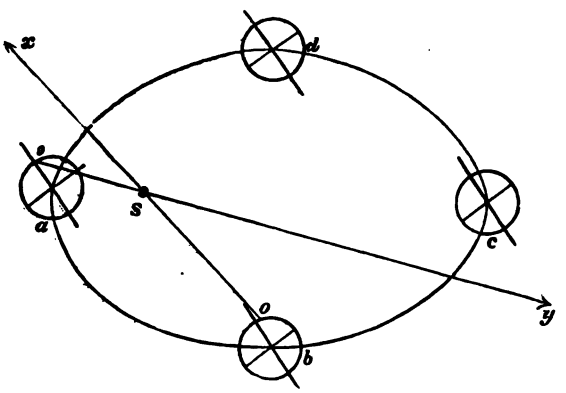
5. The zenith is a point situated on the celestial sphere vertically over the observer; his nadir is the corresponding point in the opposite hemisphere.

6. The **prime vertical** is a great circle passing through the zenith and nadir, and the east and west points of the horizon.

7. The **six o'clock hour circle** is a great circle passing through the poles, and the east and west points of the horizon.

Explanation of the Ecliptic.

Fig. 16.

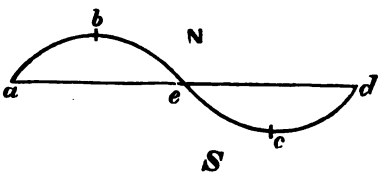


Let *a*, *b*, *c*, *d* represent the four positions of the earth, in March, June, September, and December, or at the times of the equinoxes and solstices; then an observer situated at *o*, when the Earth was at *a*, would see the sun in the direction *y*. If the Earth then travels to *b*, the Sun would appear to him to move in a contrary direction to that of the Earth's motion, or from *y* towards *x*.

If an observer were to mark the position of the Sun at the same time thus throughout the year, the path described by that body would be an ellipse (which may be regarded as a circle) cutting the celestial equator at an acute angle, the angle or inclination of the Earth's axis to the plane of its orbit, $23\frac{1}{2}°$, or more correctly 23° 27′ 45″. This figure is called the **ecliptic**, and the angle of its inclination to the equator is called the **obliquity of the ecliptic.**

If the equator be represented as a straight line, the ecliptic would be a curve, cutting it at the above angle thus,

Fig. 17.



a and *e* would represent the **equinoctial points**, and *b* and *c* the **solsticial points**. Suppose the Sun to set out from *a* on March 21st (the "vernal equinox") he would reach *b*, on June 21st (the "summer solstice" of the N hemisphere), returning to the equator at *e*, on September 21st (the autumnal equinox), then on to *c* (the winter solstice), then again to *d*, which is coincident with *a*.

The limits between *b* and *c* and the equator are called the **tropics**, extending over a belt of $23\frac{1}{2}°$ north and south of the equator. The northern limit is the **Tropic of Cancer**; the southern, the **Tropic of Capricorn.**

The Sun thus appears to move in the heavens through certain groups of stars, or constellations, lying on the ecliptic. These are twelve in number, and are called **Signs of the Zodiac.** Their names are,

Aries (the Ram)...	21st March.
Taurus (the Bull)	„ April.
Gemini (Twins)...	„ May.
Cancer (Crab)	„ June.
Leo (Lion)...	„ July.
Virgo (Virgin)	„ August.
Libra (Balances)	„ September.
Scorpio (Scorpion)	„ October.
Sagittarius (Archer)	„ November.
Capricornus (Goat)	„ December.
Aquarius (Waterbearer)	„ January.
Pisces (Fishes)	„ February.

The Sun enters each sign successively on the 21st of the month. The four periods which need to be particularly remembered are, the equinoxes (Aries and Libra), and the solstices (Cancer and Capricorn).

DEFINITIONS *(continued)*.

8. The **ecliptic** is the Sun's imaginary path amongst the fixed stars; or, a great circle cutting the equator at an angle of $23\frac{1}{2}°$.

9. The **poles of the ecliptic** are two points on the surface of the sphere, everywhere 90° distant from the ecliptic.

10. The **poles of the heavens** are those points, where the Earth's axis produced, cuts the celestial concave.

11. **Circles of altitude, vertical circles** or **circles of azimuth,** are great circles passing through the zenith and nadir, perpendicular to the horizon.

12. **Circles of declination, hour circles,** or **meridians,** are great circles passing through the poles of the heavens, perpendicular to the equator.

13. **Circles of celestial latitude** are great circles passing through the poles of the ecliptic, perpendicular to the ecliptic.

14. **Parallels of altitude** are small circles parallel to the horizon.

15. **Parallels of declination** are small circles parallel to the equator.

16. The **colures** are two meridians passing through the equinoctial and solsticial points.

17. The **twilight circle** is that parallel of altitude 18° below the horizon.

(2) *Measurements.*

18. The **altitude** of a body is an arc of a circle of altitude intercepted between the body and the horizon.

19. The **declination** of a body is the arc of a circle of declination, between the body and the equator.

20. The **latitude** of a body is the arc of a circle of latitude, between the body and the ecliptic.

21. The **right ascension** of a body is an arc of the equator, or an angle at the pole intercepted between the meridian passing over the "first point of aries," and a meridian over the object. (Measured eastward).

22. The **longitude** is an arc of the ecliptic intercepted between a circle of latitude drawn over the first point of aries, and that over the object.

23. The **amplitude** of a body is an arc of the horizon inter-

cepted between the east point and the object's rising point, or the west and its setting point.

24. **Azimuth** is an angle at the zenith, included between the meridian of the observer, and the circle of altitude passing over the body.

25. The **hour angle** is an angle at the pole included between the meridian over the observer, and that over the object.

26. **Polar distance** is an arc of a circle of declination between the body and the pole, (complement of the declination).

27. **Zenith distance** is an arc of a circle of altitude between the body and the zenith, (complement of the altitude).

28. **Colatitude** is an arc of the celestial meridian, between the zenith and pole.

(3) *Time.*

We have said that the Sun moves in a curve called the ecliptic, which appears to cut the equator at an angle of $23\frac{1}{2}°$. But this he does at an ununiform rate, passing through a different space each 24 hours, because of its elliptical form, he therefore cannot measure time accurately. To obviate this, astronomers have supposed an imaginary sun, which travels at an uniform rate on the equator. Such a body is called the **Mean Sun**, and the standard of time which it gives is that by which all clocks and watches are regulated. It is evident, that if, on the 21st March, the two Suns set out from the first point of aries, the one, by his irregular motion, will sometimes be in advance, and at other times behind, the other. Now, if two meridians be drawn, one over each Sun, the angle at the pole between them is called the **equation of time**, and is the means by which mean time, or time by the Mean Sun, may be reduced to apparent time, or that shown by the true Sun.

We should conclude, therefore, that on the 21st of March and September, or when the Suns are at aries and libra, the equation of time would become 0; but this is not the case.

The equation of time really consists of two parts; the 1st, arising from the true Sun's unequal motion in the ecliptic, and the 2nd, from its motion being in the ecliptic instead of in the equator.

So that the two parts vanish at different times, and on the days of the equinoxes, when the 1st part vanishes or becomes 0, the 2nd is left.

DEFINITIONS *(continued)*.

29. **Mean time** is the westerly hour angle or meridian distance of the Mean Sun.

30. **Apparent time** is the westerly hour angle of the apparent or true Sun.

31. **Sidereal time** is the westerly hour angle of the 1st point of aries.

32. **An apparent solar day** is the interval between two successive transits of the true Sun across the same meridian.

33. **A mean solar day** is the interval between two successive transits of the Mean Sun.

34. **A sidereal day** is the interval between two successive transits of the 1st point of aries.

35. **Equation of time** is an angle at the pole between a meridian over the true Sun, and one over the Mean Sun.

(4) *Civil and Astronomical Time.*

All ordinary clocks and watches are marked round to 12 hours. The day is divided, in common reckoning, into two parts of 12 hours each, A.M. and P.M. We say, in ordinary language, that the Sun is on the meridian at 12 o'clock, *i.e.*, 12 in the day; and from that time to 12 at midnight, the revolution made by the hour hand is called P.M. time; the remaining 12 hours being A.M. This is called **civil time.**

The problems in nautical astronomy require this distinction of A.M. and P.M. to be done away with. To obtain this, another kind of clock is called into use; its dial is marked round to 24 hours. Therefore, while the pointer of this clock makes one revolution of 24 hours, that of an ordinary clock will make two, to complete its A.M. and P.M. Suppose the Sun to come on the meridian of the observer, one clock will show 12^h; the other 24^h or 0^h. At 3 P.M., both clocks will show 3^h; at 6, 6^h; at 9, 9^h; at 12, 12^h; and after that time the distinction begins. The pointer of the clock showing civil time goes on to 1 again; but that of the astronomical clock, to 13^h. The

1^h shown by the former clock means 1^h A.M.; astronomically, it is 13^h. Now, if a solar day begins at 0^h on January 5th, for example, both clocks will agree that is January 5th till 12 at night; and after that time the first clock will show January 6th 1^h A.M.; but the astronomical one will still show January 5th 13^h. Hence results a simple rule, which it is of the greatest import to understand thoroughly before commencing any problem of nautical astronomy:—("If the time by an ordinary clock be A.M., add 12^h to it, and reckon from the preceding day; reject the A.M., and the result will be the astronomical time:") *e.g.*, August 10th, 6^h A.M. civil date is August 9th, 18^h astronomical date.

P.M. time remains as before: *e.g.*, August 8th, 5^h P.M. civil date, is August 8D 5^h astronomically.

The reason of such an arrangement is, that the elements in the Nautical Almanac are computed for astronomical date at Greenwich, and all other time must be reduced to this. In questions set for working, the time therein given is always civil date. Such time must be converted at once into astronomical date (as above) before the almanac can be used.

A mean solar day is the standard of measurement, being exactly 24 hours long; an apparent solar day is variable, on account of the true Sun's unequal motion in the ecliptic.

A sidereal day is $23^h\ 56^m$ long, measured in mean solar time, or the first point of aries comes to any meridian in an interval which is 4^m shorter than the corresponding interval marked by the Mean Sun; hence we have the equation,

$$24^h \text{ mean solar time} = 23^h\ 56^m.,$$

and on this principle, the "acceleration table," in the Nautical Almanac, is constructed, by which solar time may be converted into sidereal time, and *vice versâ*. We have,

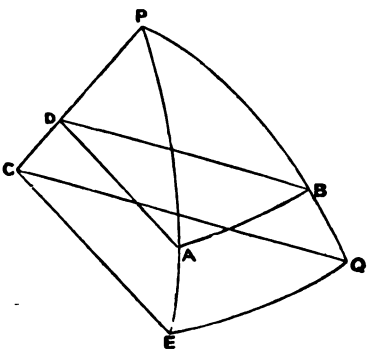
1^h solar $= 1^h\ 0^m\ 9^s{\cdot}86$ sidereal time
1^m „ $= 1^m\ 0{\cdot}16$ „ „
1^s „ $= 1{\cdot}0027$ „ „ &c.

The 1st point of aries has been observed to move 50″ eastward in the ecliptic every year. This called the "precession of the equinoxes;" and by it, in the course of centuries, the seasons would be changed.

INVESTIGATIONS OF RULES IN NAVIGATION AND NAUTICAL ASTRONOMY.

Parallel Sailing.

Fig. 18.



Let A and B be two places on the same parallel of latitude AB; let PE and PQ represent their meridians, EQ an arc of the equator, corresponding to the arc of the parallel AB. PC the polar axis of the Earth. C the centre of the earth, which is also the centre of the equator, EQ. D the centre of AB. Join DA, DB, CE, CQ.

Then, since similar arcs of circles are to each other as their radii, therefore.

AB : EQ :: DA : CE

But AD is the sine of the arc AP

and EC is the sine of EP. (Def$^{ns.}$ Circle Trig$^{y.}$)

therefore, AB : EQ :: sin AP : sin EP

But AB measures the mer. dist. between A and B.

EP = 90° and EQ is the diff. long. :

therefore, mer. dist. : D long. :: sin AP : sin 90°

But sin AP = cos AE

and AE measures the lat. of A and B

therefore mer. dist. : D long. :: cos lat. : 1

or mer. dist. = D long. cos lat. (1)

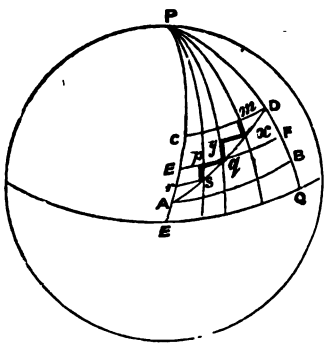
whence D long. = mer. dist. sec lat. (2)

and cos lat. $= \frac{\text{mer. dist.}}{\text{D long.}}$. (3)

which are the three expressions in use in parallel sailing.

Middle Latitude Sailing.

Fig. 19.



Let AD be a rhumb line, cutting the several meridians in the adjoining figure, *in* A, *s*, *q*, *x*, and D. Draw *xy*, *qp*, *sr*, parallel to EQ.

Then, in each of the small right angled triangles D*xm*, *xqy*, *qsp*, *s*A*r*, the departures are represented by *m*D, *yx*, *pq*, *rs*, respectively, and the differences of latitude by *mx*, *yq*, *ps*, and *r*A respectively.

It is evident that the sum of A*r*, *sp*, *yq*, *xm*, will be equal to AC, hence AC is the whole D lat. between D and A. But it will not be true that the sum of the departures, *m*D, *yx*, *pq*, *rs*, constitutes either AB or CD: for it is less than AB, and greater than CD, for the lengths of a degree of a parallel of latitude decrease as we approach the poles. But the sum of these departures will be found to be equal to a line midway between CD and EQ, or as it is termed, a departure in mid. latitude. This is the principle of Middle Latitude Sailing, and adopting the formula for Parallel Sailing, and applying it to this, we have,

Mer. dist. = D long. cos lat . . . (parallel sailing)

But mer. dist. = Dep. in mid. lat.

Hence

(1) Dep. = D long. cos middle lat.

But by the right angled triangle used in plane sailing, where dep. is the perpendicular, and dist. the hypothenuse, we have,

Dep. = dist. sin co;

hence dist. sin co. = D long. cos *m* lat.

whence

(2) D long. = dist. sin co sec *m* lat.

Again, by Plane Sailing,

$$\text{Tan Co} : = \frac{\text{Dep}}{\text{D lat.}}$$

But Dep. = D long. cos *m* lat. (1)

therefore,

$$(3)\ \text{Tan Co} : = \frac{\text{D long. cos } m \text{ lat.}}{\text{D lat.}}$$

Great Circle Sailing.

Fig. 20

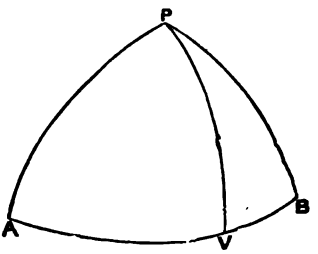
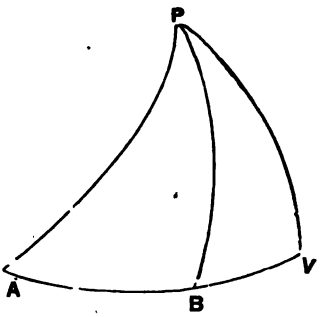


Fig. 21.



The shortest distance between any two places, is the arc of the great circle joining them. This may be proved by means of a terrestrial globe, by bringing the places to the wooden horizon, and stretching a piece of line between them, when it will be found that such a line measures the shortest distance. A succession of points on a great circle, being chosen, the latitude and longitude of each point being known, the course and distance between each is computed. If a perpendicular, in the figure, be drawn from P the pole, to AB the great circle joining A and B, it may fall either between A and B, or outside AB: the point thus obtained (V), is called the vertex, and it is the highest latitude reached in sailing over a great circle.

We have, in the figure, PA and PB, the colatitudes: the angle APB measures the D long. between A and B, and APV between A and V: PV is the colatitude of V.

It is required to find an expression for the distance, and for the latitude and longitude of the vertex.

Distance.

In the spherical triangle APB, we have,

$$\text{Cos AB (dist)} = \cos \text{PA} \cdot \cos \text{PB} + \sin \text{PA} \sin \text{PB} \cos \text{P}$$

$$\text{or cos dist} = \cos b \cdot \cos a + \sin b \cdot \sin a \cos \text{P}$$

$$\text{But cos P} = (2 \cos^2 \tfrac{1}{2} \text{P} - 1).$$

$$\text{therefore cos dist} = \cos a \cos b + \sin a \sin b \cdot (2 \cos^2 \tfrac{1}{2} \text{P} - 1)$$

$$\textit{i. e.}, \text{ cos dist} = \cos a \cos b - \sin a \sin b + 2 \sin a \sin b \cos^2 \tfrac{1}{2} \text{P}$$

$$= \cos (a+b) + 2 \sin a \sin b \cos^2 \tfrac{1}{2} \text{P}$$

Subtracting both sides from 1.

$$1 - \cos \text{dist} = 1 - \cos (a + b) - 2 \sin a \sin b \cos {}^2\tfrac{1}{2} \text{P}$$

$$\text{or } 2 \sin^2 \frac{\text{dist}}{2} = 2 \sin^2 \frac{(a + b)}{2} - 2 \sin a \sin b \cos {}^2\tfrac{1}{2} \text{P}$$

Let $\sin a \sin b \cdot \cos {}^2\tfrac{1}{2} \text{P} = \sin {}^2\theta$

$$\text{then } \sin^2 \frac{\text{dist}}{2} = \sin^2 \frac{a + b}{2} - \sin {}^2\theta$$

$$\text{whence } \sin^2 \frac{\text{dist}}{2} = \sin \left(\frac{a + b}{2} + \theta\right) \sin \left(\frac{a + b}{2} - \theta\right)$$

$$\text{and } \sin \frac{\text{dist}}{2} = \sqrt{\sin \left(\frac{a + b}{2} + \theta\right) \sin \left(\frac{a + b}{2} - \theta\right)}$$

Latitude of Vertex.

In the triangle APB,

$$\frac{\text{Sin AB}}{\text{Sin PB}} = \frac{\text{Sin P}}{\text{Sin A}}$$

$$\text{therefore } \sin \text{AB} \cdot \sin \text{A} = \sin \text{PB} \cdot \sin \text{P}$$

$$\text{and } \sin \text{A} = \frac{\sin \text{PB} \sin \text{P}}{\sin \text{AB}}$$

Multiply by sin AP

$$\text{then } \sin \text{A} \cdot \sin \text{AP} = \frac{\sin \text{PB} \cdot \sin \text{PA} \cdot \sin \text{P}}{\sin \text{AB}}$$

But since APV is a right angled spherical triangle,

$$\text{Sin A} \cdot \sin \text{AP} = \sin \text{PV}$$

$$\text{and, } \sin \text{PV} = \frac{\sin \text{PA} \cdot \sin \text{PB} \cdot \sin \text{P}}{\sin \text{AB}}$$

$$\text{whence } \cos \text{lat V} = \frac{\cos \text{lat A} \cdot \cos \text{lat B} \cdot \sin \text{D long.}}{\sin \text{dist.}}$$

Longitude of Vertex.

In the right angled spherical triangle APV,

$$\text{Cos APV} = \cot \text{PA} \cdot \tan \text{PV}$$

i. e., cos D long from A to V = tan lat A cot lat V and applying long of A to this, that of V is obtained.

The D long is called east or west, as B (the place of greater latitude) is east or west of A.

Second Method for the Distance.

$$\text{Cos dist} = \cos a \cos b + \sin a \sin b \,.\, \cos \text{P}$$
$$\text{and } \cos \text{P} = (1 - 2 \sin^2 \tfrac{1}{2} \text{P})$$

therefore

$$\begin{aligned}\text{Cos dist} &= \cos a \cos b + \sin a \sin b\,(1 - 2 \sin^2 \tfrac{1}{2} \text{P}) \\ &= \cos a \cos b + \sin a \sin b - 2 \sin a \sin b \sin^2 \tfrac{1}{2} \text{P} \\ &= \cos (a - b) - 2 \sin a \sin b \,.\, \sin^2 \tfrac{1}{2} \text{P}\end{aligned}$$

Adding 1 to both sides,

$$1 + \cos \text{dist} = 1 + \cos (a - b) - 2 \sin a \sin b \sin^2 \tfrac{1}{2} \text{P}$$

Let $\sin a \sin b \sin^2 \tfrac{1}{2} \text{P} = \cos^2 \theta$

$$2 \cos^2 \tfrac{1}{2} \text{dist} = 2 \cos^2 \frac{a - b}{2} - 2 \cos^2 \theta$$

then, since $\cos^2 x - \cos^2 y = \sin (y + x) \,.\, \sin (y - x)$

$$\cos^2 \tfrac{1}{2} \text{dist} = \sin \left\{ \theta + \frac{a - b}{2} \right\} \sin \left\{ \theta - \frac{a - b}{2} \right\}$$

and

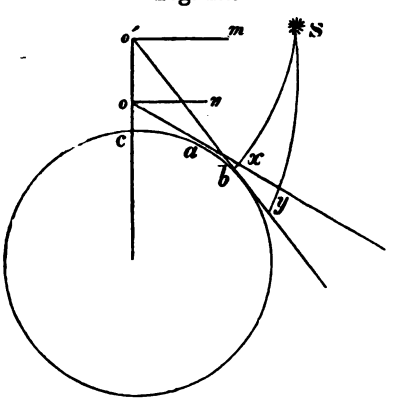
$$\cos \tfrac{1}{2} \text{dist} = \sqrt{\sin \left\{ \theta + \frac{a - b}{2} \right\} \sin \left\{ \theta - \frac{a - b}{2} \right\}}$$

where a and b are the colatitudes, and P the diff. long.

Explanation of the Corrections for an Altitude.

(1) *Dip.*

Fig. 22.

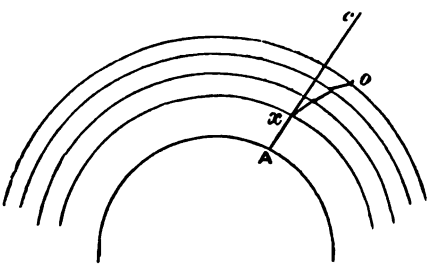


If an altitude were taken from the surface of the sea, the eye of the observer would be upon his horizon, and no correction for dip would be used. But as he is generally elevated some feet above it when measuring an object's altitude, the altitude thus obtained would differ from that taken from the surface of the water. In the figure the observer is represented as being at o; ox is a line touching the earth, drawn from his eye, and is his horizon.

The altitude of the body S, measured in such a position, would be Sx: that, in a higher position, o' would be Sy. The angles *noa* and *mo'b*, measure the dip of the horizon respectively. Now, as the observer at o' is higher than at o, it is evident that the angle *mo'b* is greater than *noa*, and the altitude Sy is greater than Sx. Hence the greater the elevation above the sea level, the greater the altitude measured, and therefore the dip must be subtracted from an altitude.

(2) *Refraction.*

Fig. 23.



When a ray of light passes through a dense medium it is bent in direction as it approaches an observer's eye.

In the figure, the ray from o is gradually bent towards the perpendicular until when i reaches A the body would appear in the direction of the last ray Ax, viz., Axc. The effect of refraction is to increase an altitude, and it must be subtracted from an observed altitude.

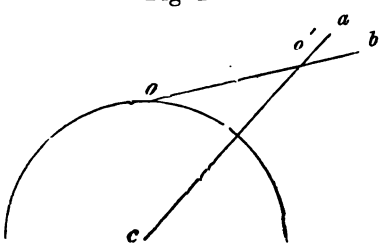
(3) *Semi-diameter.*

This is a correction to be applied to the altitudes of the Sun or Moon, or of a planet. The altitude either of the upper or lower limb is observed, and hence, to obtain the altitude of the object's centre, if the former be taken, the semi-diameter is subtracted, if the latter, added.

(4) *Parallax.*

A correction, to reduce an altitude as observed from the surface of the earth, to what it would be, if taken from the centre.

Fig. 24.



In the figure, an object at *o*, would be seen in a higher position from C, the centre of the earth, (viz., C *a*), than if viewed from *o'* at the surface, (*o' b*).

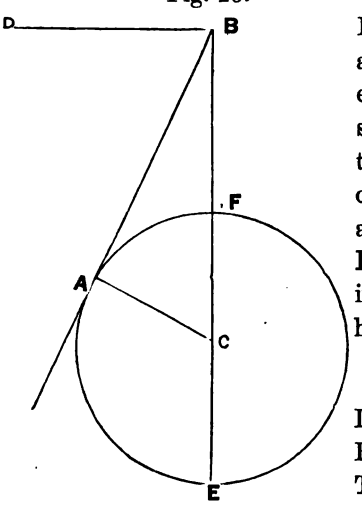
Hence the correction called parallax must be added to the altitude observed. It may be defined, as the angle at the body's centre, subtended by the earth's radius, (*o o' c* in the figure).

The parallax of a body when rising or setting, is called its "horizontal parallax." This correction vanishes, as the body approaches the Zenith.

INVESTIGATION OF THE RULES IN NAUTICAL ASTRONOMY.

An expression for the Dip.

Fig. 25.



Let AFE represent the Earth: B, the place of an observer: DB a horizontal line drawn from his eye: BA a tangent to the Earth's surface at A. Join CA. Then the line BF is the height of the observer's eye above the horizon, and the angle DBA is called the Dip or depression of his horizon in that position (BA), below the horizontal line.

To prove that BCA = the Dip.

Because DBC = 90°

Therefore DBA + ABC = 90°

Again, since BAC = 90°
therefore ABC + ACB = 90°
and · ABC + ACB = DBA + ABC
Omit ABC, then,
ACB = DBA.
But DBA measures the dip
Hence ACB = the dip
In the Investigation, let
BF = h : CF = CA = r, then,
In the right angled triangle BAC,

$$\text{Cos BCA} = \frac{\text{AC}}{\text{CB}} = \frac{r}{r+h}$$

$$\text{and } 1 - \cos \text{dip} = 1 - \frac{r}{r+h} = \frac{r+h-r}{r+h} = \frac{h}{r+h}$$

$$\text{But } \frac{h}{r+h} = \frac{h}{r} \text{ nearly,}$$

for h at the most is only a few feet, and r the Earth's radius = 4,000 miles: hence $r + h$ may be said to be nearly equal to r.

$$\text{Therefore, } 1 - \cos \text{dip} = \frac{h}{r} \text{ nearly,}$$

$$\text{or } 2 \sin^2 \frac{\text{dip}}{2} = \frac{h}{r}$$

Multiplying both sides by 2,

$$4 \sin^2 \frac{\text{dip}}{2} = \frac{2h}{r} = \frac{2}{r} \times h$$

Taking the square root,

$$2 \sin \frac{\text{dip}}{2} = \sqrt{\frac{2}{r}} \times \sqrt{h}$$

But the sine of a small arc is nearly equal to the arc itself, multiplied by the sin of 1′, therefore,

$$\text{Sin } \frac{\text{dip}'}{2} = \frac{\text{dip}'}{2} \cdot \sin 1'$$

$$\text{and } 2 \sin \frac{\text{dip}'}{2} = \text{dip}' \cdot \sin 1' \cdot$$

Substituting,

$$\text{Dip}' \cdot \sin 1 = \sqrt{\frac{2}{r}} \times \sqrt{h}.$$

$$\text{and dip}' = \frac{\sqrt{\frac{2}{r}} \times \sqrt{h}}{\sin 1'} *$$

But the fraction, $\frac{\sqrt{\frac{2}{r}}}{\sin 1'}$ may be rejected, being small, and

$$\text{Therefore Dip}' = \sqrt{h} \text{ nearly,}$$

which expression will give the dip in cases where no great accuracy is required: *e. g.*, if an observer be raised 36 feet above the sea, the dip for an altitude in that position is $\sqrt{36} = 6'$.

From this arises a useful rule, viz.:—

To find the distance of the visible horizon at sea.

$$\text{Rule; dist:} = \sqrt{h + \tfrac{1}{2}h}$$

If an observer be raised 24 feet above the sea, he can notice a vessel just appearing in sight, at a distance of 6 miles from him;

$$\text{for dist,} = \sqrt{24 + 12} = 6 \text{ miles,}$$

and the diameter of his horizon would be 12 miles

The following proof verifies this:—

In the preceding figure, by (Euc. III. 36),

$$\text{EB} \cdot \text{BF} = \text{BA}^2$$

$$i.e., (\text{EF} + \text{FB}) \cdot \text{BF} = \text{BA}^2$$

$$\text{Let FB} = h \text{: and BA} = d\text{:}$$

$$\text{then } (2r + h) \cdot h = d^2$$

$$\text{or } 2rh + h^2 = d^2$$

But h^2 is so small that it may be omitted,

$$\text{then, } 2rh = d^2$$

$$\text{But } r = 4{,}000 \text{ miles nearly,}$$

$$\text{and } \frac{h}{5{,}280} \text{ represents } h \text{ as the fraction of a mile,}$$

$$\text{therefore, } 8{,}000 \times \frac{h}{5{,}280} = d^2$$

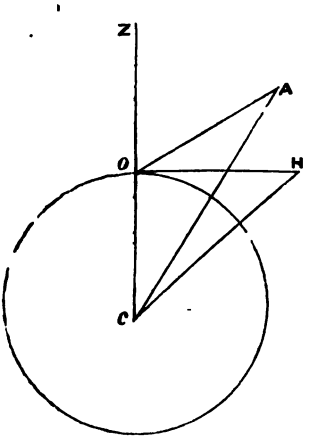
$$\text{or } \frac{3h}{2} \text{ (nearly)} = d^2$$

$$\text{whence } d = \sqrt{h + \tfrac{1}{2}h}, \text{ (nearly).}$$

* The "Table of Dip" is computed from this expression.

PARALLAX.

Fig. 26.



Let Z be the zenith: OH the horizon of an observer at O. A the position of an object: then the angle OHC is called its **horizontal parallax**, and OAC its **parallax in altitude**: and AOH measures its **apparent altitude.**

In the right angled triangle, OCH,

$$\frac{OC}{CH} = \sin OHC = \sin \text{hor. par}^{x.}$$

In the oblique angled triangle OCA,

$$\frac{OC}{CA} = \frac{\sin OAC}{\sin COA}$$

But sin COA = sin ZOA (being supplements),

and sin ZOA = cos AOH (being complements),

hence, $\frac{OC}{CA} = \frac{\sin OAC}{\cos AOH} = \frac{\sin \text{par}^{x.} \text{ in alt}}{\cos \text{app}^{t.} \text{ alt}}$.

and $\frac{OC}{CA} = \frac{OC}{CH}$: for CA = CH.

$$\text{therefore, sin hor par}^{x.} = \frac{\sin \text{par}^{x.} \text{ in alt}}{\cos \text{app}^{t.} \text{ alt}}$$

$$\text{or sin hor par}^{x.} \times \cos \text{app}^{t.} \text{ alt} = \sin \text{par}^{x.} \text{ in alt.}$$

But par$^{x.}$ in alt and hor par$^{x.}$ are both small arcs, hence,

sin hor par$^{x.}$″ = hor par$^{x.}$″ sin 1″

and sin par$^{x.}$ in alt″ = par$^{x.}$ in alt″ sin 1″

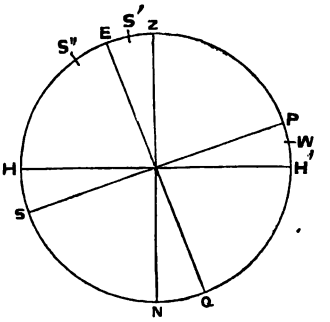
therefore, hor par$^{x.}$″ sin 1″ × cos alt = par$^{x.}$ in alt″ sin 1″

whence, par$^{x.}$ in alt″ = hor par$^{x.}$″ cos app$^{t.}$ alt.

From this formula it is manifest that hor par$^{x.}$ is greater than par$^{x.}$ in alt, the former being diminished by cos alt.

Meridian Altitudes.

Fig. 27.



Let the circle represent the celestial meridian, then Z is the zenith: P is the elevated pole: EQ the equator: HH′ the horizon. Let S′, S″ and W be the three respective positions of an object on the meridian.

1st. To prove that

The altitude of the pole above the horizon measures the latitude of the place of observation.

Since ZH′ = 90°: and EP = 90°

therefore ZH′ = EP

i. e., ZP + PH′ = EZ + ZP

Omit ZP

then PH′ = ZE

But since, on the Earth, the arc of a meridian between the observer and the equator measures his latitude, therefore the corresponding arc in the heavens (ZE) will measure the same, hence PH′ measures the latitude.

2nd. Let an object be on the meridian at S', in which case its zenith distance and declination are measured the same way, *i. e.*, both N or S. We have, then,

$$ZE = \text{latitude}$$
$$ZS' = \text{zenith distance}$$
$$ES' = \text{declination}$$
$$\text{and } ZE = ZS' + S'E$$
$$\text{or lat} = \text{zen dist} + \text{dec}^{n}.$$

For if P be the N pole, the zenith will be north of the object, and the declination will be measured northwards; and *vice versâ*.

3rd. Let an object be on the meridian at S'', where the zen dist and declination are of contrary names, then,

$$ZE = \text{latitude}$$
$$ZS'' = \text{zenith distance}$$
$$ES'' = \text{declination}$$
$$\text{and } ZE = ZS'' - S''E$$
$$\text{or lat} = \text{zen dist} - \text{decl}^{n}.$$

In this case, supposing P to be the north pole, the zenith would be north of the object, and the declination would be measured south.

4th. Let the object be below pole as at W, then,

$$PH' = \text{latitude}$$
$$PW = \text{polar distance}$$
$$WH' = \text{altitude}$$
$$\text{and } PH' = PW + WH'$$
$$\text{or lat} = \text{polar dist} + \text{alt.}$$

The Sun's greatest declination, north or south, never exceeds 23°.28', and therefore he never approaches either pole nearer than 66°.32', or recedes from it farther than 113°.28'.

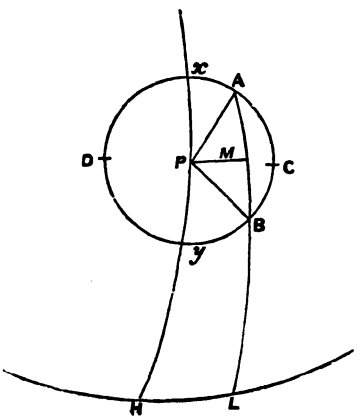
From this it appears that the lowest latitude in which the Sun is ever below pole, (*i. e.*, the one in which he is seen to touch the horizon without setting), is 66°.32'. This would take place at the time of the solstices in either hemisphere. Circumpolar objects are those whose polar distances are less than the latitude PH', and

which therefore make their revolutions round the pole without setting, and are visible on the meridian, twice in 24 hours.

Latitude by Pole Star.

We have seen that the altitude of the pole measures the latitude. If, therefore, an object were placed exactly at the pole, its altitude (corrected for index error, dip, &c.) would measure the latitude. The pole star is a circumpolar object, which makes its revolution at $1\frac{1}{2}^{\circ}$ distance from the pole. A correction, therefore, must be applied to its altitude to give that of the pole. In the adjoining figure,

Fig. 28.

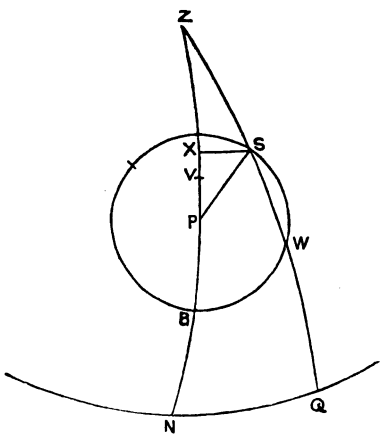


PH represents the meridian, A and B two positions of the star, one above and the other below the pole P. C and D two in which the star is on a level with the pole. When the star is at x, the question becomes one of meridian altitudes, and all corrections vanish. As the arc PA revolves with the star round the point P, it marks an angle x PA, which varies in size, with the position of the star. At x the angle is 0. When the star is at C it is 6^h; at y, 12^h; at D 18^h. As the arc PH, the alt of the pole is the object sought, when the star is at C or at D (*i. e.*, when its meridian distance or hour angle is 6^h or 18^h), its altitude is nearly equal to the latitude PH, and in rough calculations, may be taken for it. If the star be in the position A, its altitude, AL, is too great for PH by the quantity, AM, and the correction AM is subtracted from the altitudes. This is the case in any position on the upper half of the circle. On the contrary, if the star be at B, its altitude, BL, is too little for PH by the quantity MB, which must be added. Hence this practical rule:

If the hour angle is between 6^h and 18^h, the correction is additive to; in any other case it must be subtracted from the altitude.

INVESTIGATION.

Fig. 29.



Let Z be the zenith, P the pole, SWB the diurnal circle described by the pole star. Let S be the position of the star when observed; ZSQ a circle of altitude passing over it. Join PS, and from S draw SX perpendicular to ZN. Then in the spherical triangle Z X S, because ZXS is a right angle, therefore ZSX is less than a right angle, and ZS is greater than ZX. Therefore a circle described with centre Z and distance ZS, would cut below X; suppose it to cut in V. Then PX is the first correction to be applied to the altitude, and VX the second.

First Correction.

The triangle XPS being small, may be considered a plane triangle; for the star is situated only at $1\frac{1}{2}°$ from the pole, and that portion of the sphere may be considered a plane surface, when compared with the whole curvature.

Hence XP = PS . cos XPS

or 1st cor. = polar dist × cos hour angle.

,, $= p''$ cos h.

Second Correction.

By construction,

$$ZS = ZV = (ZX + XV)$$

and cos ZS = cos (ZX + XV)

or cos ZS = cos ZX . cos XV − sin ZX . sin XV

But since ZXS is a right angled spherical triangle, we have, by Napier's rules,

$$\cos ZS = \cos ZX \,.\, \cos XS\,.$$

therefore $\cos ZX \,.\, \cos XS = \cos ZX \cos XV - \sin ZX \sin XV$

But VX being small, $\cos VX = 1$ nearly,

hence, $\cos ZX \cos XS = \cos ZX - \sin ZX \sin XV\,.$

whence, $\sin ZX \sin XV = \cos ZX - \cos ZX \cos XS$

$$= \cos ZX \,.\, (1 - \cos XS)$$

$$= \cos ZX \,.\, 2 \sin^2 \frac{XS}{2}$$

$$\text{and } \sin VX = \cot ZX \,.\, 2 \sin^2 \frac{XS}{2}$$

But $\sin VX'' = VX'' \,.\, \sin 1''\,.$

and $\cot ZX = \tan XN = \tan SQ = \tan \text{alt}$

$$\text{also } \sin \frac{XS''}{2} = \frac{XS''}{2} \,.\, \sin 1''$$

$$\text{and } \sin^2 \frac{XS}{2} = \frac{XS^2}{4} \sin^2 1''$$

$$2 \sin^2 \frac{XS}{2} = \frac{XS^2}{2} \,.\, \sin^2 1''$$

But $XS = SP \,.\, \sin XPS = (p \,.\, \sin h)\,.$

therefore $XS^2 = (p \sin h)^2\,.$

$$\text{therefore } 2 \sin^2 \frac{XS}{2} = \frac{(p \sin h)^2}{2} \,.\, \sin^2 1''$$

Substituting these values,

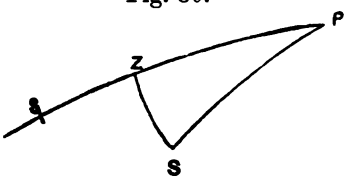
$$VX'' \,.\, \sin 1'' = \tan \text{alt} \,.\, \frac{(p \sin h)^2}{2} \,.\, \sin^2 1''$$

whence, VX (2nd cor.) $= \frac{1}{2} \,.\, \sin 1'' \,.\, \tan \text{alt} \,.\, (p \sin h)^2\,.$

which is always added to the altitude.

Reduction to the Meridian.

Fig. 30.



Let S be the position of an object when near the meridian, and S′ its place when on the meridian, and suppose that in the interval its polar distance does not change. On this supposition the problem of "Reduction to the Meridian" depends. Let PZ the colatitude $= l$: PS the polar dist $= p$: ZS the zenith distance at the time of observation, $= z$; and ZS′ the meridian zenith distance of the object $= z'$; and the

angle ZPS the object's hour angle $= h$. Then the reduction is the correction to be applied to the altitude when at S, to convert it into a meridian altitude, and the correction may be defined, as the difference between ZS and ZS', or between z and z'. By supposition, we have,

$$\begin{aligned} \text{PS} &= \text{PS}' \\ \text{or } p &= z' + l' \\ \text{and } z' &= p - l' \ . \\ \text{and } \cos z' &= \cos (p - l) \\ \text{or } \cos z' &= \cos p \cos l' + \sin p \sin l' \end{aligned}$$

But in the spherical triangle ZPS,

$$\cos z = \cos p \cos l' + \sin p \sin l' \ . \cos h$$

Subtracting the second equation from the first,

$$\begin{aligned} \cos z' - \cos z &= \sin p \ . \sin l' \ . (1 - \cos h) \\ &= \sin p \ . \sin l' \ . 2 \sin^2 \tfrac{1}{2} h \end{aligned}$$

$$\text{or } 2 \sin \frac{z + z'}{z} \ . \ \sin \frac{z - z'}{z'} = \sin p \ . \sin l' \ . 2 \sin^2 \tfrac{1}{2} h \ .$$

But the meridian zenith distance may be assumed nearly equal to the zenith distance at the time of observation, therefore

$$z = z' \text{ and } \frac{z + z'}{2} = z' \text{ nearly.}$$

$$\text{also } \frac{z - z'}{2} = \frac{\text{reduction}}{2} \text{ ; and substituting,}$$

$$2 \sin z' \ . \sin \frac{r}{2} = \sin p \ . \sin l' \ . 2 \sin^2 \tfrac{1}{2} h$$

$$\text{But } \sin \frac{r''}{2} = \frac{r''}{2} \ . \sin 1''$$

$$\text{and } 2 \sin \frac{r''}{2} = r'' \ . \sin 1''$$

therefore,

$$\sin z' \ . r'' \ . \sin 1'' = \sin p \sin l' \ . 2 \sin^2 \tfrac{1}{2} h$$

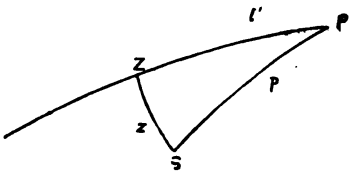
$$\text{or sin mer zen dist } . r'' \sin 1'' = \cos \text{ dec } . \cos \text{ lat } . 2 \sin^2 \tfrac{1}{2} h.$$

$$\text{whence } r'' = \cos \text{ dec cos lat } . \text{ cosec MZD } . \frac{2 \sin^2 \tfrac{1}{2} h}{\sin 1''}$$

In the case of the Sun, the meridian altitude is the greatest altitude during the day, and the reduction must always be added to any altitude of that body, to obtain his meridian altitude.

Formulæ for Computing an Object's Hour Angle.

Fig. 31.



Altitude taken for the purpose of computing "Time," should be observed when the body is far from the meridian, or as near as possible to the "Prime Vertical;" for then, any error in the altitude will produce a less effect in the hour angle deduced from it than at another time. But on the contrary an error in an altitude taken near the meridian, produces a great effect upon the hour angle. Such altitudes, carefully observed, are suitable for the computation of latitude, as in the case of "reduction to the meridian."

In the figure, let

$$\text{PZ} = l' : \text{PS} = p : \text{ZS} = z : \text{ZPS} = h .$$

Then,

$$\cos h = \frac{\cos z - \cos l' . \sin p}{\sin l' . \sin p}$$

But $\cos z = \sin \text{alt}$
and $\cos l' = \sin \text{lat} .$
„ $\sin l' = \cos \text{lat} .$

$$\text{and } \cos h = \frac{\sin \text{alt} - \sin \text{lat} . \cos p}{\cos \text{lat} . \sin p .}$$

Subtract both sides from 1 . ,

$$\text{then } 1 - \cos h = 1 - \left(\frac{\sin a - \sin l . \cos p}{\cos l . \sin p}\right)$$

$$= \frac{\cos l . \sin p - \sin a + \sin l . \cos p}{\cos l . \sin p .}$$

$$= \frac{\sin (p + l) - \sin a}{\cos l . \sin p}$$

$$= \frac{2 \cos \tfrac{1}{2} (p + l + a) . \sin \tfrac{1}{2} (p + l - a)}{\cos l . \sin p}$$

Let $\frac{1}{2} (p + l + a) = \text{S}$
then $\frac{1}{2} (p + l - a) = (\text{S} - a)$
and $1 - \cos h = 2 \sin^2 \frac{1}{2} h .$

$$\text{therefore, } 2 \sin^2 \tfrac{1}{2} h = \frac{2 \cos \text{S} . \sin (\text{S} - a)}{\cos l . \sin p}$$

$$\text{and } \sin \tfrac{1}{2} h = \sqrt{\sec l . \operatorname{cosec} p . \cos \text{S} . \sin (\text{S} - a)}$$

To Deduce the Time from an Hour Angle.

If the Sun be the object observed, the hour angle (if westerly) will measure the apparent time; but if the Sun be East of the meridian at the time, the hour angle must be taken from 24^h to make it westerly. And, as in all problems where longitude is concerned the mean time is required, this may be found, by applying the equation of time to the apparent time. But in the case of any other object being observed, the following rule is to be universally adopted, viz.:—

True mean time at place = obj^{t} west^{r} h^{r} angle, + obj^{t} right as^{n} — mean Sun's right as^{n} (increasing by 24^h if necessary).

Longitude.

By the longitude of a place, is understood the arc of the Earth's equator, between the meridian of Greenwich and the meridian of the place; or, which is the same thing, the angle at the pole between those meridians. The Earth turns on its axis once in 24 hours from west to east, which motion, to a spectator on the Earth, causes the Sun to appear to move in a contrary direction, and to make a circuit round the Earth in that interval. The equator is the only great circle which runs in an east and west direction on the Earth's surface, and therefore in 24 hours, the 360° of it will be passed over by the Sun. Hence in 1 hour, 15° will be passed over by that body; in other words 15° = 1 hour. We have,

1 hour = 15° . 0′ . 0″
1 min. = 0 . 15′ . 0″
1 sec. = 0 . 0′ . 15″

This the connection between time and longitude.

Now as the apparent motion of the Sun is from east to west, the day will begin in east longitude before it will commence in west; or places east of Greenwich, will have later time than Greenwich; and at places west, the time will be earlier.

If therefore we are possessed of two time keepers, one showing the exact time at the place in which we are, and the other showing the exact time at Greenwich, at this instant at the place, the difference of these times will be the longitude of the place. If the time at the place be the later of the two, we then know we are east

of Greenwich; if **earlier, west** of it. The following examples illustrate this:—

Ex. 1.—A clock shows 5^h (Astronomical time) at a place, and the time at Greenwich, at that instant, is 8^h; What is the longitude?

$$8^h - 5^h = 3^h = 45°, \text{ (since } 1^h = 15°)$$

and the place is **west** of Greenwich, since the time is **earlier** at place.

Ex. 2.—Greenwich time 18^h: at place 22^h; difference $4^h = 60°$ E, because the time at the place is **later.**

Ex. 3.—In longitude 60° 15′ W, the time is 8 P.M. January 7th; Find the Greenwich time?

	d.	h.	m.	s.		long.
Here time at place = Jan.	7	8	0	0		60° 15′W.
Longitude (in time) =	+	4	1	0		4
Greenwich time	7	12	1	0	6)24	1 0
						4^h 1^m 0^s

Ex. 4.—The time at Greenwich is December 6th, 7 A.M.; Find the time at a place in longitude 81° 30′W?

	d.	h.	m.	s.
Astronomical date at Greenwich Dec.	5	19	0	0
Longitude (in time)		5	26	0
Time at place (earlier than Greenwich)	5	13	34	0

Hence the following rules in use in Nautical Astronomy,

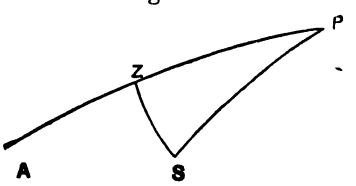
1. To find Greenwich date, **add, west** longitude, to astronomical time at place; **subtract, east.**

2. To find longitude, take the astronomical difference between the time at place and that at Greenwich, and then,

"If the Greenwich time be **least,**
The longitude is **east.**
If the Greenwich time be **best,**
The longitude is **west.**"

To Compute an Object's Azimuth.

Fig. 32.



Let S be the place of an object in the heavens, then PZS or AZS is its "azimuth."

Let PS $= p$: PZ $= l'$: ZS $= z$; then,

$$\cos \text{PZS} = \frac{\cos p - \cos l' \cdot \cos z}{\sin l' \cdot \sin z}$$

But $\cos \text{PZS} = -\cos \text{AZS}$, (being supplements),

and $\cos l' = \sin \text{lat } (l)$

$\cos z = \sin \text{alt } (a)$

$\sin l' = \cos \text{lat}$

$\sin z = \cos \text{alt}$

Substituting,

$$-\cos \text{AZS} = \frac{\cos p - \sin l \cdot \sin a}{\cos l \cdot \cos a}$$

Add 1 to both sides, then

$$1 - \cos \text{AZS} = 1 + \frac{\cos p - \sin l \cdot \sin a}{\cos l \cdot \cos a}$$

$$= \frac{\cos l \cos a + \cos p - \sin l \sin a}{\cos l \cdot \cos a}$$

$$= \frac{\cos (a + l) + \cos p}{\cos l \cdot \cos a}$$

$$= \frac{2 \cos \tfrac{1}{2}(a + l + p) \cdot \cos \tfrac{1}{2}(a + l - p)}{\cos l \cdot \cos a}$$

Then, if $\frac{1}{2}(a + l + p) = \text{S}$

$\frac{1}{2}(a + l - p) = (\text{S} - p)$

and $1 = \cos \text{AZS} = 2 \sin^2 \frac{1}{2} \text{AZS}$,

therefore, $2 \sin^2 \frac{1}{2} \text{AZS} = \frac{2 \cos \text{S} \cdot \cos (\text{S} - p)}{\cos a \cdot \cos l}$

or $\sin \frac{1}{2} \text{AZS} = \sqrt{\sec a \sec l \cdot \cos \text{S} \cdot \cos (\text{S} - p)}$

and the difference between the observed azimuth, and the computed, is the "variation of the compass."

Without the Altitude.

When the altitude is not observed, we have the polar dist p, and the colatitude l', and the hour angle h. Then by Napier's Analogies,

$$\text{Tan } \tfrac{1}{2}(Z + S) = \cos \tfrac{1}{2}(p - l') \sec \tfrac{1}{2}(p + l') \cot \tfrac{1}{2} h$$
$$\text{and tan } \tfrac{1}{2}(Z - S) = \sin \tfrac{1}{2}(p - l') \operatorname{cosec} \tfrac{1}{2}(p + l') \cot \tfrac{1}{2} h$$

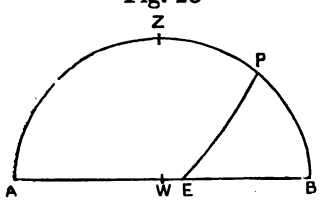
from which angle Z is obtained,

and the supplement of it is AZS, the true azimuth.

If the polar distance p be greater than l', then the angle Z is greater than S: but if l' the colatitude, be greater than p, S is greater than Z. In the first of these cases, the azimuth Z, $= \frac{1}{2}(Z + S) - \frac{1}{2}(Z - S)$, but in the latter, Z, $= \frac{1}{2}(S + Z) - \frac{1}{2}(S - Z)$.

For the Amplitude of an Object.

Fig. 23



Let AZB represent the meridian: Z the zenith: P the pole: E the place of a body rising or setting: W the east or west point of the horizon. Then EW measures its amplitude: PB the latitude: PE the polar distance of the object. Then by Napier's Rules for right angled spherical triangles,

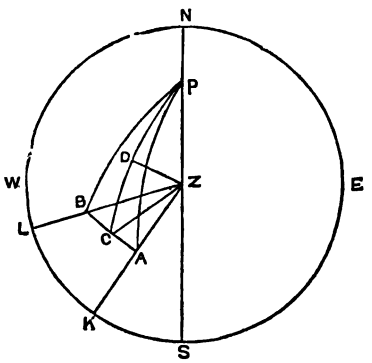
$$\cos PE = \cos PB \,.\, \cos BE$$
$$\text{or } \cos p = \cos \text{lat} \,.\, \cos BE$$
$$\text{and } \cos BE = \sin EW$$
$$= \sin \text{amp}$$
$$\text{then } \cos p = \cos \text{lat} \,.\, \sin \text{amp}$$
$$\text{or sin dec} = \cos \text{lat sin amp}$$
$$\text{whence sin amp} = \sin \text{dec} \,.\, \sec \text{lat}$$

If a body rises either to the north or south of the equator (which passes through the east and west points), it will travel during the day on that side of the equator. Hence if a body has north declination, it will rise in the north east quarter of the compass, and if south, in the south east.

Therefore the amplitude is of the **same name** as the **declination**, while bearings and azimuths depend upon the **latitude**, the former being of the **same name**, and the latter **contrary** to the latitude.

*Latitude by Ivory's Method of Double Altitudes.**

Fig 34.



This method, called from its inventor, "Ivory's Method," is of use in those cases where the declination of the body changes very slowly: or in the cases of the Sun and Moon, when those bodies reach their greatest northern or southern declination, and the hourly change is small. The problem is generally rejected when the observer is in the tropics.

Let B and A be the two positions of the object observed: Z the zenith; P the pole; NS the observer's meridian; NLKS the horizon; ZL and ZK two circles of altitude passing over the places of the body; PA, PB circles of declination. Then ZA and ZB measure its zenith distances; AK and BL its altitudes at the times of observation; and PA and PB, the polar distances, are considered equal which is the grand supposition of the method.

Bisect the angle BPA by PC, which bisects BA in C. Join ZC, and draw ZD perpendicular to PC.

Then the angle BPA, in the case of the object being the Sun, is the elapsed apparent time between the observations, AK $= a$: BL $= b$: the angle APC (which is the half elapsed time) $= h$.

Then AC is called Arc 1; CP, Arc 2; ZD, Arc 3; DC, Arc 4; DP, Arc 5; and ZP, the colatitude (the object sought).

For Arc 1.

In the right angled spherical triangle, ACP,

$$\text{Sin AC} = \text{sin AP} \cdot \text{sin APC}$$

or sin arc 1 = sin *p.d.* sin HET.

For Arc 2.

$$\cos \text{AP} = \cos \text{AC} \cdot \cos \text{CP}$$

or cos *p.d.* = cos arc 1. cos arc 2.

whence cos *p.d.* sec arc 1 = cos arc 2.

* Modified by the late Edward Riddle, Esq., F.R.A.S., Head Master of the Nautical School, Greenwich Hospital.

For Arc 3.

In the spherical triangles ZCA, ZCB,

$$\cos ZA = \cos ZC \cdot \cos CA + \sin ZC \cdot \sin CA \cos ZCA$$

$$\text{and } \cos ZB = \cos ZC \cdot \cos CB + \sin ZC \cdot \sin CB \cdot \cos ZCB$$

But $\cos ZA = \sin AK = \sin a$

„ $\cos ZB = \sin BL = \sin b$.

and $CA = CB$

and $\cos ZCA = \sin ZCD$ (being complements)

„ $\cos ZCB = -\cos ZCA = -\sin ZCD$

Therefore, substituting,

$$\sin a = \cos ZC \cdot \cos CA + \sin ZC \cdot \sin CA \cdot \sin ZCD$$

$$\sin b = \cos ZC \cdot \cos CA - \sin ZC \cdot \sin CA \cdot \sin ZCD$$

Substracting,

$$\sin a - \sin b = 2 \sin ZC \cdot \sin CA \cdot \sin ZCD$$

But $\sin ZC \cdot \sin ZCD = \sin ZD$

for ZDC is a right angle

therefore,

$$2 \cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2}(a - b) = 2 \sin CA \cdot \sin ZD$$

or $\cos \frac{1}{2}$ sum of alts $\times$ $\sin \frac{1}{2}$ diff $=$ sin arc 1 . sin arc 3

whence

$$\cos S \cdot \sin D \cdot \operatorname{cosec} \text{arc } 1 = \sin \text{arc } 3.$$

For Arc 4.

Again, adding the two equations,

$$\sin a + \sin b = 2 \cos ZC \cdot \cos CA$$

But in the right angled triangle ZDC,

$$\cos ZC = \cos ZD \cdot \cos DC$$

therefore,

$$2 \sin S \cdot \cos D = 2 \cos ZD \cdot \cos DC \cdot \cos CA$$
$$= 2 \cos \text{arc } 3 \cdot \cos \text{arc } 4 \cdot \cos \text{arc } 1 .$$

and, $\sin S \cdot \cos D \cdot \sec \text{arc } 1 \cdot \sec \text{arc } 3 = \cos \text{arc } 4$

For Arc 5.

$$CP = CD + DP$$

$$\text{or arc } 2 = \text{arc } 4 + \text{arc } 5$$

$$\text{and arc } 2 - \text{arc } 4 = \text{arc } 5$$

For the Latitude.

In the right angled triangle PDZ,

$$\cos PZ = \cos PD \cdot \cos DZ$$

$$\text{or } \sin \text{lat} = \cos \text{arc } 5 \cdot \cos \text{arc } 3.$$

Sumner's Method of Double Altitudes.

This method was published by Captain Sumner, of the United States Navy, in 1842.

One half of the Earth is always turned towards a celestial object; therefore, if we join the positions of those places where the body appears on the horizon, the result will be a great circle, the pole of which has the object in the zenith. At all places lying on a small circle parallel to this, the altitude of the body will be the same; such a circle is called a "Circle of Equal Altitudes," or a "Circle of Position."

When an observation of a heavenly body is taken, the observer must be upon some circle of equal altitudes; and after the body has changed its bearing considerably, and consequently its altitude, he will be upon another. The intersection of these two circles will give his position. This is the principle of Sumner's method. If we take only small portions of the arcs of such circles, we have straight lines intersecting, and these are called "Lines of Position." For the practical working of the method, let two latitudes be assumed, one on each side of the latitude by account. We have also the altitudes and polar distances of the object, at each observation; then the process is as follows:—

With 1st alt: 1st assumed lat: and 1st polar dist: work a longitude, . . (a)

With 1st alt: 2nd assumed lat: and 1st polar dist: work a longitude, (a')

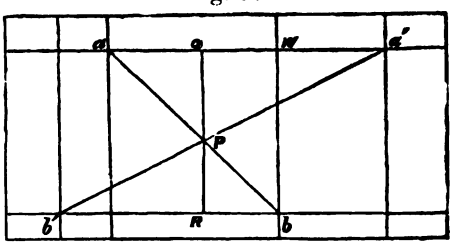
With 2nd alt: 1st assumed lat: and 2nd polar dist: work a longitude, . . (b)

With 2nd alt: 2nd assumed lat: and 2nd polar dist: work a longitude . . (b')

Now, if the four positions (assumed) of the ship be joined, two together, viz., a and b, and a' and b', the intersection of these lines will be the observer's true place.

INVESTIGATION.

Fig. 35



Let the four positions thus obtained be laid down on a chart; a and b joined give the line of position for the 1st altitude, and a' and b' for the 2nd. Draw a meridian QR through P, the intersection of $a\,b$ and $a'\,b'$

Then PQ is the correction for latitude, measured from $a\,a'$: and a Q the correction for longitude.

Let the diff long from a to b, viz., a W, = C

„ the diff of assumed lats: QR = D

„ the greater diff long given by either latitude $a\,a'$ = A

„ the less diff long given by either latitude $b\,b'$ = B

Then, in the similar triangles, a P a' and b P b',

$$bb' : aa' :: \text{PR} : \text{PQ}$$

$$\text{hence } \frac{bb'}{aa'} = \frac{\text{PR}}{\text{PQ}} \ldots \quad (1)$$

Add 1 to both sides, then,

$$\frac{bb'}{aa'} + 1 = \frac{\text{PR}}{\text{PQ}} + 1$$

$$\text{or } \frac{bb' + aa'}{aa'} = \frac{\text{PR} + \text{PQ}}{\text{PQ}}$$

$$\text{or } \frac{\text{B} + \text{A}}{\text{A}} = \frac{\text{RQ}}{\text{PQ}} = \frac{\text{D}}{\text{PQ}}$$

inverting,

$$\frac{\text{A}}{\text{A} + \text{B}} = \frac{\text{PQ}}{\text{D}} = \frac{\text{cor : for lat}}{\text{D}}$$

therefore,

$$\text{cor for lat} = \frac{\text{A} \times \text{D}}{\text{A} + \text{B}}$$

Again, in the similar triangles a QP and a W b

$$a\text{Q} : \text{QP} :: a\text{W} : \text{W}b$$

$$\text{or } \frac{a\text{Q}}{\text{QP}} = \frac{a\text{W}}{\text{W}b}$$

$$\text{or } a\text{Q} = \frac{a\text{W} \times \text{QP}}{\text{W}b} = \frac{\text{C} \times \text{QP}}{\text{QR}} = \frac{\text{C} \times \text{QP}}{\text{D}}$$

But, as above,

$$\text{QP (cor for lat)} = \frac{A \times D}{A + B}$$

$$\text{hence } a\text{Q (cor for long)} = \frac{C \times A \times D}{D \times (A + B)}$$

$$\text{or cor for long} = \frac{A \times C}{A + B}$$

It may happen that a, a', b and b' so fall that when $a\,b$ and $a'\,b'$ are joined, they must be produced in order to intersect, in which case, the observer's position is outside the parallels $a\,a'$ and $b\,b'$. Then the difference in the proof is, that in equation (1), both sides are taken from unity, then,

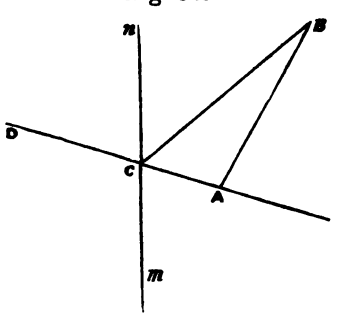
$$\text{cor for lat} = \frac{A \times D}{A - B}$$

$$\text{and cor for long} = \frac{A \times C}{A - B}.$$

Correction for Ship's Run.

The object of this is, to reduce the altitude of any object observed at one place, to what it would have been, if observed at the same instant, at another place.

Fig. 36.



Suppose CB to be the distance run by the ship in the interval between the observations, and mCA the bearing of the Sun at the 1st observation. From B draw BA perpendicular to CA, then CA is the correction for ship's run. The angle BCA is included between the direction of the ship's run and Sun's bearings, and

$$\text{CA} = \text{CB} \times \cos \text{BCA}$$

$$\text{or cor} = \text{dist run} \times \cos \text{BCA}$$

This correction in minutes of arc is + to the 1st alt if BCA be less than eight points or a right angle; if greater, the perpendicular BA will fall on CD, then,

$$\cos \text{BCA} = -\cos \text{BCD}$$

and the correction must be subtracted from the 1st altitude

The dist run (CB) is found by multiplying the no. of miles run per hour, by the no. of hours in the interval.

Direct Method of Double Altitudes.

Fig. 37.

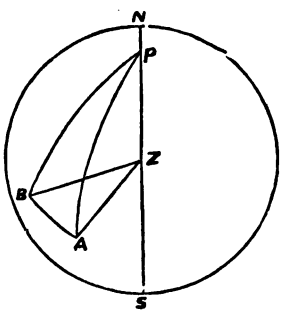
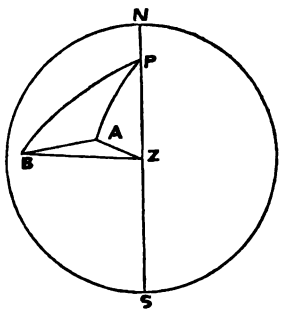


Fig. 38.



This method depends immediately upon formulæ of Spherical Trigonometry already proved. Two observations of a heavenly body are taken; the zenith and polar distances are known, and the polar angle between the two positions of the object. The computation of the latter is the only difficulty in the problem. The rules for this are as follows:—

1. If the Sun be the object taken,

The polar angle is the interval of apparent time between the observations. The interval shown by the chronometer is mean time. If therefore the equation of time be applied to each of the chronometer times (+ to mean time), the result will be the interval in apparent time, which must be reduced to arc.

2. If two stars be observed.

If the observations be simultaneous, the difference of their right ascensions is the polar angle. But should there be an interval, the difference between the times shown by the chronometer must be converted into sidereal time, by the "Acceleration Table," (Nautical Almanac). This is called the "sidereal interval;" it must be added to the RA of the first observed star, and then the difference between this and the RA of the second star is the polar angle, to be reduced to arc.

The computation is then as follows:—

Fig. (37). Where PA and PB, the polar distances, are greater than PZ, the colatitude of the place.

(1) In the triangle PBA; to find BA.

$$\text{Sin}^2\,\theta = \sin \text{BP} \cdot \sin \text{PA} \cdot \cos^2 \tfrac{1}{2}\,\text{P}$$

$$\text{Sin}^2\,\frac{\text{AB}}{2} = \sin\left\{\frac{\text{PB} + \text{PA}}{2} + \theta\right\} \cdot \sin\left\{\frac{\text{PB} + \text{PA}}{2} - \theta\right\}$$

(2) In the same triangle; to find $\frac{1}{2}$ PBA.

$$\text{Cos}\,\tfrac{1}{2}\,\text{PBA} = \sqrt{\sin \text{S} \cdot \sin(\text{S} - \text{PA}) \cdot \text{cosec PB} \cdot \text{cosec BA}}$$

(3) In the triangle ZBA; to find $\frac{1}{2}$ ZBA.

$$\text{Cos}\,\tfrac{1}{2}\,\text{ZBA} = \sqrt{\sin \text{S} \cdot \sin(\text{S} - \text{ZA}) \cdot \text{cosec ZB} \cdot \text{cosec BA}}$$

(4) $\frac{1}{2}$ PBA $-$ $\frac{1}{2}$ ZBA $=$ $\frac{1}{2}$ PBZ

(5) In the triangle PBZ; to find PZ.

$$\text{Sin}^2\,\theta = \sin \text{PB} \cdot \sin \text{BZ} \cdot \cos^2 \tfrac{1}{2}\,\text{PBZ}$$

$$\text{Sin}^2\,\frac{\text{PZ}}{2} = \sin\left\{\frac{\text{PB} + \text{BZ}}{2} + \theta\right\} \cdot \sin\left\{\frac{\text{PB} + \text{BZ}}{2} - \theta\right\}$$

and (90 $-$ PZ) $=$ latitude.

Fig. (38). Where the polar distances are less than PZ, the colatitude.

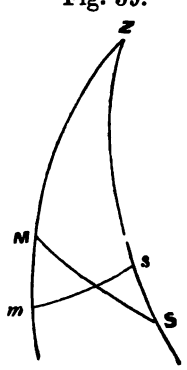
The only difference lies in (4)

Where $\frac{1}{2}$ PBA $+$ $\frac{1}{2}$ ZBA $=$ $\frac{1}{2}$ PBZ

The longitude is obtained with the altitude of the farthest star: latitude, and polar distance.

Formula for Clearing a Lunar Distance from the Effects of Parallax and Refraction.

Fig. 39.



Let Z be the zenith: ZMm and ZsS, two vertical circles passing over the positions of the Moon and Sun, (or star).

I. In the figure, suppose

M to represent the true place of the Moon,
m „ „ apparent place of the Moon,
S „ „ true place of the Sun,
s „ „ apparent place of the Sun.

Then, since the Moon's parallax is always greater than her refraction, therefore M is above m. (See Article on Parallax and Refraction). And for an opposite reason, s is above S. Join ms, and MS.

II. In the investigation, let

M° = the true altitude of the Moon,
m° = the apparent altitude of the Moon,
S° = the true altitude of the Sun,
s° = the apparent altitude of the Sun,
MS = D, (the true distance of centres),
ms, = d, (the apparent distance of centres).

Then, we know that in any spherical triangle ABC,

$$\text{Cos}^2\tfrac{1}{2}\text{A} = \sin\tfrac{1}{2}(b+c+a)\sin\tfrac{1}{2}(b+c-a)\,\text{cosec}\,b\,.\,\text{cosec}\,c$$

Apply this formula to the triangle Zms, then,

$$\text{Cos}^2\tfrac{1}{2}\text{Z} = \sin\tfrac{1}{2}(Zm+Zs+ms)\sin\tfrac{1}{2}(Zm+Zs-ms)\,\text{cosec}\,Zm\,\text{cosec}\,Zs$$

But Zm = zen dist of m = $(90 - m^\circ)$
and Zs = ... s = $(90 - s^\circ)$
„ $ms = d$.

Substituting,

$$\text{Cos}^2\tfrac{1}{2}\text{Z} = \sin\tfrac{1}{2}\left\{\overline{90-m}+\overline{90-s}+d\right\}\sin\tfrac{1}{2}\left\{\overline{90-m}+\overline{90-s}-d\right\}$$
$$\text{cosec}\,(90-m)\,\text{cosec}\,(90-s)$$

$$= \sin\tfrac{1}{2}\left\{180-\overline{m+s-d}\right\}\sin\tfrac{1}{2}\left\{180-\overline{m+s+d}\right\}\sec m\,\sec s$$

$$= \sin\left\{90-\tfrac{1}{2}(m+s-d)\right\}.\sin\left\{90-\tfrac{1}{2}(m+s+d)\right\}.\sec m\,\sec s$$

$$= \cos\tfrac{1}{2}(m+s-d).\cos\tfrac{1}{2}(m+s+d).\sec m\,.\,\sec s.$$

Let $\tfrac{1}{2}(m+s+d) = X$
Then $\tfrac{1}{2}(m+s-d) = (X-d)$
Then, $\cos^2\tfrac{1}{2}Z = \cos X\,.\,\cos(X-d)\,.\,\sec m\,\sec s.$

In the triangle ZMS, let

$$\text{Sin}^2\theta = \sin ZM\,.\,\sin ZS\,.\,\cos^2\tfrac{1}{2}Z$$

Then

$$\text{Sin}^2\frac{MS}{2} = \sin\left\{\frac{ZM+ZS}{2}+\theta\right\}\sin\left\{\frac{ZM+ZS}{2}-\theta\right\}$$

or $\sin^2\theta = \cos M\,.\,\cos S\,.\,\cos^2\tfrac{1}{2}Z$

But $\cos^2\tfrac{1}{2}Z = \cos X\cos(X-d)\,.\,\sec m\,\sec s$

therefore,

$$\text{Sin}^2\theta = \cos M\cos S\,.\,\cos X\,.\,\cos(X-d)\,.\,\sec m\,\sec s.$$

and $$\sin^2\frac{D}{2} = \sin\left\{\frac{ZM+ZS}{2}+\theta\right\}\sin\left\{\frac{ZM+ZS}{2}-\theta\right\}$$

Now,

$$\left\{\frac{ZM+ZS}{2}+\theta\right\}=\left\{\frac{90-M+90-S}{2}+\theta\right\}=\left\{\frac{180-\overline{M+S}}{2}+\theta\right\}$$
$$=\left\{90-\left(\frac{M+S}{2}-\theta\right)\right\}$$

and, in like manner,

$$\left\{\frac{ZM+ZS}{2}-\theta\right\}=\left\{\frac{90-M+90-S}{2}-\theta\right\}=\left\{\frac{180-\overline{M+S}}{2}-\theta\right\}$$
$$=\left\{90-\left(\frac{M+S}{2}+\theta\right)\right\}$$

Substituting, we have,

$$\text{Sin}^2\frac{MS}{2}=\sin\left\{90-\left(\frac{M+S}{2}-\theta\right)\right\}.\sin\left\{90-\left(\frac{M+S}{2}+\theta\right)\right\}$$

$$\text{or } \sin^2\frac{D}{2}=\cos\left(\frac{M+S}{2}-\theta\right).\cos\left(\frac{M+S}{2}+\theta\right).$$

The following method may be used with a table of versines (as given in the appendix to Inman's Navigation, p. 267). Let the true zen distances ZM and ZS be denoted by z and z', and the two apparent altitudes (their complements) by a and a'. Let the true distance be represented by x, and the apparent by y.

Then in the triangle ZMS,

$$\text{Cos Z}=\frac{\cos \text{MS}-\cos \text{MZ}.\cos \text{ZS}}{\text{Sin MZ}.\sin \text{ZS}}$$
$$=\frac{\cos x-\cos z.\cos z'}{\sin z.\sin z'}$$

and in the triangle ZmS,

$$\text{Cos Z}=\frac{\cos ms-\cos mZ.\cos Zs}{\sin mZ.\sin Zs}$$
$$=\frac{\cos y-\sin a.\sin a'}{\cos a.\cos a'}$$

therefore,

$$\frac{\text{Cos } x-\cos z\cos z'}{\sin z.\sin z'}=\frac{\cos y-\sin a.\sin a'}{\cos a\cos a'}$$

Add 1 to both sides, then,

$$1+\frac{\cos x-\cos z\cos z'}{\sin z.\sin z'}=\frac{\cos y-\sin a.\sin a'}{\cos a\cos a'}+1.$$

or,

$$\frac{\text{Sin } z.\sin z'+\cos x-\cos z\cos z'}{\sin z\sin z'}=\frac{\cos a\cos a'+\cos y-\sin a\sin a'}{\cos a\cos a'}$$

but $\sin z \,.\, \sin z' - \cos z \cos z' = -\cos(z + z')$

and $\cos a \cos a' - \sin a \sin a' = \cos(a + a')$

therefore,

$$\frac{\text{Cos } x - \cos(z + z')}{\sin z \,.\, \sin z'} = \frac{\cos y + \cos(a + a')}{\cos a \cos a'}$$

then,

$$\left\{\text{Cos } x - \cos(z+z')\right\} \cos a \cos a' = \left\{\cos y + \cos(a+a')\right\} \sin z \sin z'$$

or,

$$\left\{\text{Cos } x - \cos(z+z')\right\} = \left\{\cos y + \cos(a+a')\right\} . \frac{\sin z \,.\, \sin z'}{\cos a \,.\, \cos a'}$$

Transposing,

$$\text{Cos } x = \cos(z + z') + \left\{\cos y + \cos(a+a')\right\} . \frac{\sin z \,.\, \sin z'}{\cos a \,.\, \cos a'}$$

$$\text{Let } \frac{\sin z \,.\, \sin z'}{\cos a \cos a'} = 2 \cos \text{A}$$

then,

$$\text{Cos } x = \cos(z+z') + \left\{\cos y + \cos(a+a')\right\} . 2 \cos \text{A}$$

$$= \cos(z+z') + 2 \cos y \,.\, \cos \text{A} + 2 \cos(a + a') \,.\, \cos \text{A}$$

But by Plane Trigonometry,

$2 \cos y \cos \text{A} = \cos(y + \text{A}) + \cos(y - \text{A})$

and $2 \cos(a + a') \cos \text{A} = \cos(a + a' + \text{A}) + \cos(a + a' - \text{A})$

therefore,

$$\text{Cos } x = \cos(z+z') + \cos(y+\text{A}) + \cos(y-\text{A}) + \cos(a+a'+\text{A}) + \cos(a+a'-\text{A})$$

but $\text{vers } x = 1 - \cos x$

or $\cos x = 1 - \text{vers } x$

then,

$$1 - \text{vers } x = \Big(1 - \text{vers}(z+z')\Big) + \Big(1 - \text{vers}(y + \text{A})\Big) + \Big(1 - \text{vers}(y - \text{A})\Big) + \Big(1 - \text{vers}(a + a' + \text{A})\Big) + \Big(1 - \text{vers}(a + a' - \text{A})\Big)$$

i. e.,

$$-\text{vers } x = -\text{vers}(z + z') + \Big(1 - \text{vers}(y + \text{A})\Big) + \Big(1 - \text{vers}(y - \text{A})\Big) + \Big(1 - \text{vers}(a + a' + \text{A})\Big) + \Big(1 - \text{vers}(a + a' - \text{A})\Big)$$

changing the signs,

$$\text{vers } x = \text{vers } (z - z') + \text{vers } (y + A) + \text{vers } (y - A) + \text{vers } (a + a' + A) + \text{vers } (a + a' - A) - 4$$

A Third Method of Investigation *

For the True Distance

In the triangle Zms,

$$\text{Cos Z} = \frac{\cos ms - \cos Zs \cdot \cos Zm}{\sin Zs \cdot \sin Zm}$$

or $\cos ms - \cos Zs \cdot \cos Zm = \sin Zs \cdot \sin Zm \cdot \cos Z$.

Add $\sin Zs \cdot \sin Zm$ to both sides, then,

$$\cos ms - \cos Zs \cos Zm + \sin Zs \cdot \sin Zm = \sin Zs \cdot \sin Z m \cos Z + \sin Zs \cdot \sin Z m$$

then

$$\sin Zs \cdot \sin Zm - \sin Zs \cdot \sin Zm \cdot \cos Z = \sin Zs \cdot \sin Zm + \cos Zs \cos Zm - \cos ms$$

$$= \cos (Zs - Zm) - \cos ms$$

$$= 1 - \cos ms - 1 + \cos (Zs - Zm)$$

or

$$\sin Zs \cdot \sin Zm (1 - \cos Z) = 1 - \cos ms - \left\{1 - \cos (Zs - Zm)\right\}$$

but $1 - \cos = \text{vers}$

therefore, $\sin Zs \cdot \sin Zm \cdot \text{vers Z} = \text{vers } ms - \text{vers } (Zs - Zm)$

and $\sin Zs \cdot \sin Zm : 1 :: \text{vers } ms - \text{vers } (Zs - Zm) : \text{vers Z}$

It may in like manner be shown, that, in the triangle ZMS,

$\sin ZS \cdot \sin ZM : 1 :: \text{vers MS} - \text{vers } (ZS - ZM) : \text{vers Z}$

whence,

$$\sin Zs \cdot \sin Zm : \sin ZS \cdot \sin ZM :: \text{vers } ms - \text{vers } (Zs - Zm) : \text{vers MS} - \text{vers } (ZS - ZM)$$

then,

$$\frac{\sin ZS \cdot \sin SM}{\sin Zs \cdot \sin Zm} = \frac{\text{vers MS} - \text{vers } (ZS - ZM)}{\text{vers } ms - \text{vers } (Zs - Zm)}$$

and

$$\text{vers MS} - \text{vers} (ZS - ZM) = \left\{\text{vers } ms - \text{vers} (Zs - Zm)\right\} \cdot \frac{\sin ZS \cdot \sin ZM}{\sin Zs \cdot \sin Zm}$$

or

$$\text{vers MS} = \left\{\text{vers } ms - \text{vers } (Zs - Zm)\right\} \cdot \frac{\sin ZS \cdot \sin ZM}{\sin Zs \cdot \sin Zm} + \text{vers } (ZS - ZM)$$

* Andrew Mackay, LL.D., F.R.S., "on Longitude."

$$\text{Let } \frac{\sin ZS \,.\, \sin ZM}{\sin Zs \,.\, \sin Zm} = \bar{d}$$

then,

$$\text{vers } MS = \left\{ \text{vers } ms - \text{vers } (Zs - Zm) \right\} . \bar{d} + \text{vers } (ZS - ZM)$$

$$\text{and let, } \left\{ \text{vers } ms - \text{vers } (Zs - Zm) \right\} . \bar{d} = a$$

$$\text{then, vers } MS = \text{vers } (ZS - ZM) + a.$$

Fourth Method of Clearing the Lunar Distance.

In the figure where

m = ☾'s apparent altitude
s = *'s " "
M = ☾'s true altitude
S = *'s " "

we have

$$\text{Cos}^2 \tfrac{1}{2} z = \frac{\cos \frac{1}{2} (m + s + ms) \cos \frac{1}{2} (m + s - ms)}{\cos m \cos s}$$

$$= \frac{2 \cos \frac{1}{2} (m + s + ms) \cos \frac{1}{2} (m + s - ms)}{2 \cos m \cos s}$$

$$= \frac{\cos (m + s) + \cos ms}{\cos (m + s) + \cos (m - s)}$$

$$= \frac{1 + \cos (m + s) - 1 + \cos ms}{1 + \cos (m + s) - 1 + \cos (m - s)}$$

$$= \frac{1 - \cos (180 - \overline{m + s}) - (1 - \cos ms)}{1 - \cos (180 - \overline{m + s}) - \left(1 - \cos (m - s)\right)}$$

$$= \frac{\text{vers } (180 - \overline{m + s}) - \text{vers } ms}{\text{vers } (180 - \overline{m + s}) - \text{vers } (m - s)}$$

Let vers $(180 - \overline{m + s}) = A$
vers ms . $= D$
vers $(m - s)$ $= B$

then,

$$\text{Cos}^2 \tfrac{1}{2} Z = \frac{A - D}{A - B}$$

or $A - D : A - B :: \cos^2 \frac{1}{2} Z : 1$

In the same way, it can be shown that

$$\text{Cos}^2 \tfrac{1}{2} Z = \frac{\text{vers } (180 - \overline{M + S}) - \text{vers } MS}{\text{vers } (180 - \overline{M + S}) - \text{vers } (M - S)}$$

$$\text{Let vers } (180 - \overline{M + S}) = a$$
$$\text{vers MS} = d$$
$$\text{vers } (M - S) = b$$

then,

$$\text{Cos}^2\tfrac{1}{2} Z = \frac{a - d}{a - b}, \text{ and,}$$

$$a - d : a - b :: \cos^2\tfrac{1}{2} Z : 1$$

But

$$A - D : A - B :: \cos^2\tfrac{1}{2} Z : 1$$

therefore,

$$A - B : A - D :: a - b : a - d.$$

$$\text{or } a - d = \frac{(A - D)(a - b)}{(A - B)}$$

whence

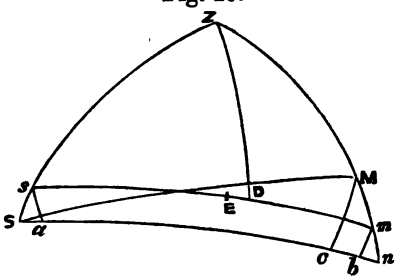
$$d = a - \frac{(A - D)(a - b)}{(A - B)}$$

Fifth Method of Clearing the Lunar Distance.

(1) To Clear the Distance from Refraction.

In the figure, let *sm* be the apparent distance, S*n* the distance cleared of Refraction,* and SM the true distance. From Z draw ZD perpendicular to *sm*, and bisect *sm* in E. From *s* draw S*a* perpendicular to S*n*; and from M and *m*, draw M*c*, and *mb* perpendicular to S*n*.

Fig. 40.



Then in the spherical triangle Z*sm*, we have given the zenith distances, Z*s*, and Z*m*, and the apparent distance *ms*, from which the angles Z*sm*, and Z*ms*, may be found, thus,

$$\text{Cos}\tfrac{1}{2}\text{Z}sm = \sqrt{\sin\tfrac{1}{2}(\text{Z}s + sm + \text{Z}m)\sin\tfrac{1}{2}(\text{Z}s + sm - \text{Z}m)\,\text{cosec}\,\text{Z}s\,\text{cosec}\,sm}$$

and

$$\text{Cos}\tfrac{1}{2}\text{Z}ms = \sqrt{\sin\tfrac{1}{2}(\text{Z}m + ms + \text{Z}s)\sin\tfrac{1}{2}(\text{Z}m + ms - \text{Z}s)\,\text{cosec}\,\text{Z}m\,\text{cosec}\,ms}$$

The angles at the Moon and other object being thus obtained, the corrections for refraction are found as follows:—

*s*S and *mn* represent the "refractions in altitude," and S*a* and *nb* the corrections for refraction, to be applied to the apparent distance, *sm*, to give the "parallactic distance."*

S*s* and *mn* can be taken from a Table of Refraction (*e.g.* Riddle, No. III), with apparent altitudes of the Moon and other object as arguments, then, the angles Z*ms*, and Z*n*S, and Z*sm*, and Z*Sn* being taken as equal, we have

$$S\alpha = Ss \cdot \cos s \quad S\alpha = R \cos Zsm$$

$$\text{and } nb = mn \cos mnb = R^1 \cos Zms$$

where R and R^1 represent the "refractions in altitude"

$$\text{then, } Sn = sm + R \cos Zsm + R^1 \cos Zms.$$

Should the angle Z*sm* or Z*ms* be found to exceed 90°, the term R cos Z*sm*, or R^1 cos Z*ms*, will be negative, and the signs must be changed, and the corrections applied accordingly.

(2) To clear the Distance from Parallax.

By Spherical Trigonometry, we have, in the triangle *s*Z*m*,

$$\text{Cos } sZ : \cos Zm :: \cos sD : \cos Dm.$$

or,

$$\text{Cos } sZ + \cos Zm : \cos sZ - \cos Zm :: \cos sD + \cos Dm : \cos sD - \cos Dm$$

but

$$\text{Cos } sZ + \cos Zm : \cos sZ - \cos Zm :: \cot \frac{sZ+Zm}{2} : \tan \frac{sZ-Zm}{2}$$

And, similarly,

$$\text{Cos } sD + \cos Dm : \cos sD - \cos Dm :: \cot \frac{sD+Dm}{2} : \tan \frac{sD-Dm}{2}$$

hence,

$$\text{Cot } \frac{sZ+Zm}{2} : \tan \frac{Zs-Zm}{2} :: \cot \frac{sm}{2} : \tan \frac{sD-Dm}{2}$$

or,

$$\frac{\text{Tan } Zs+Zm}{2} \cdot \frac{\cot Zs+Zm}{2} : \tan \frac{Zs+Zm}{2} \cdot \tan \frac{Zs-Zm}{2} :: \tan \frac{Sm}{2} \cdot \cot \frac{Sm}{2}$$

$$: \tan \frac{sD+Dm}{2} \cdot \tan \frac{sD-Dm}{2}$$

* Called the "Parallactic Distance," being affected by parallax only.

But tan . cot = 1

therefore,

$$1 : \tan\frac{Zs+Zm}{2} . \tan\frac{Zs-Zm}{2} :: 1 : \tan\frac{sD+Dm}{2} . \tan\frac{sD-Dm}{2}$$

or,

$$\text{Tan}\frac{Zs+Zm}{2} . \tan\frac{Zs-Zm}{2} = \tan\frac{sD+Dm}{2} . \tan\frac{sD-Dm}{2}$$

or,

$$\text{Tan}\frac{Sm}{2} : \tan\frac{sZ+Zm}{2} :: \tan\frac{sZ-Zm}{2} : \frac{\tan sD-Dm}{2}$$

i. e.,

$$\text{Tan}\frac{sD+Dm}{2} : \tan\frac{sZ+Zm}{2} :: \tan\frac{sZ-Zm}{2} : \tan\frac{sD-Dm}{2}$$

But

$$\frac{sD+Dm}{2} = sE ; \frac{sD-Dm}{2} = DE \text{ (or arc 1)}$$

therefore

Tan DE (arc 1) = cot ½ sum app alts . tan ½ diff* app alts . cot ½ app dist

and ½ dist + DE (arc 1) = *s*D (arc 2) and D*m* (arc 3)

* ½ diff alts being = ½ diff zenith distances.

In the right angled spherical triangle Z*s*D,

$$\cos Zs D = \cot Zs . \tan sD$$

and in the small triangle *s*S*a*, cos *s*S*a*, $= \frac{Sa}{Ss}$

angles Z*s*D and *s*S*a* being taken as equal.

therefore,

$$\tan sD . \cot sZ = \frac{Sa}{sS} \text{ or } Sa = \frac{sS . \tan sD}{\tan sZ}$$

But *s*S the corr^{n} of ⊙'s alt for ref^{n} = 57″ . tan Z*s* nearly

therefore *sa* = 57″ . tan *s*D (arc 2)

And, in a similar manner for the two triangles Z*m*D and *mbn*, it may be proved that,

nb the corr^{n} of ☾'s alt for ref^{n} = 57″ . tan D*m* (arc 3)

Again, in the triangles ZD*m* and M*cn*

$$\cos ZmD = \tan Dm . \cot Zm .$$

$$\text{and } \cos MnC \text{ (or } ZmD) = \frac{nc}{nM}$$

therefore

$$\frac{nc}{nM} = \tan Dm . \cot Zm ; \text{ or } nc = nM \tan Dm \cot Zm$$

But Mn or par^{x} in alt = hor par^{x} sin Zn.
and Zm may be taken equal to Zn

therefore

nc = hor par^{x} tan Dm cos Zn = hor par^{x} tan arc 3 . sec ☾'s app alt

Hence,

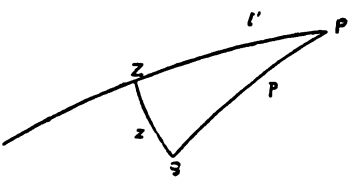
Por^{l} log par^{x} in dist = por^{l} log hor par^{x} + log cot arc 3 + log cosec ☾'s app alt
× ×

× When the proportional logs are used in combination with other logs, the factors require to be inverted.

For Computing an Object's Altitude.

First Method.

Fig. 41.



The elements we require to know before computing an altitude, are the polar distance of the objects; the colatitude of the place of observation; and the hour angle or meridian distance of the body. This, in the case of the object being the Sun, is the true apparent time at place: in that of any other object we must apply the rule,—

Mean time at place + mean Sun's RA − object's RA
= object's hour angle.

In the figure, let PZ = l' (colat)
" " PS = p (polar dist)
" " ZS = z (the zen dist)
and the angle ZPS = h (the hour angle)
then z is the element required, and

$$\cos z = \cos p \cos l' + \sin p \sin l' . \cos h$$
but $\cos h = (2 \cos^2 \tfrac{1}{2}h - 1)$

Then,

$$\cos z = \cos p \cos l' + \sin p \sin l' . (2 \cos^2 \tfrac{1}{2}h - 1)$$
$$= \cos p \cos l' - \sin p \sin l' + 2 \sin p \sin l' . \cos^2 \tfrac{1}{2}h$$
$$= \cos (p + l') + 2 \sin p \sin l' \cos^2 \tfrac{1}{2}h$$
Let $\sin p \sin l' \cos^2 \tfrac{1}{2}h = \sin^2 \theta$

and subtract both sides from 1, then,

$$1 - \cos z = 1 - \cos (p + l') - 2 \sin^2 \theta$$

$$\text{or } 2 \sin^2 \frac{z}{2} = 2 \sin^2 \frac{p + l'}{2} - 2 \sin^2 \theta$$

$$\text{whence } \sin \frac{z}{2} = \sqrt{\sin \left(\frac{p + l'}{2} + \theta \right) \sin \left(\frac{p + l'}{2} - \theta \right)}.$$

Second Method.

As before,

$$\cos z = \cos p \cos l' + \sin p \sin l' \cos h$$

But $\cos h = 1 - 2 \sin^2 \frac{1}{2}h$

$$\cos z = \cos p \cos l' + \sin p \sin l' (1 - 2 \sin^2 \tfrac{1}{2}h)$$
$$= \cos p \cos l' + \sin p \sin l' - 2 \sin p \sin l' \sin^2 \tfrac{1}{2}h$$
$$= \cos (p - l') - 2 \sin p \sin l' \sin^2 \tfrac{1}{2}h$$

Add 1 to both sides, then,

$$1 + \cos z = 1 + \cos (p - l') - 2 \sin p \sin l' \sin^2 \tfrac{1}{2}h$$

Let $\sin p \,.\, \sin l' \sin^2 \frac{1}{2}h = \cos^2 \theta$

then, $2 \cos^2 \frac{z}{2} = 2 \cos^2 \frac{p - l'}{2} - 2 \cos^2 \theta$

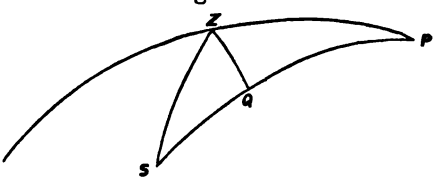
But it can be shown, that

$$\cos^2 p - \cos^2 q = \sin (q + p) \,.\, \sin (q - p)$$

therefore, $\cos \frac{z}{2} = \sqrt{\sin \left(\theta + \frac{p - l'}{2}\right) \sin \left(\theta - \frac{p - l'}{2}\right)}$

Third Method.

Fig. 42.



Draw ZQ perpendicular to PS, then by Napier's rules for right angled spherical triangles, we have,

(1) Sin ZQ = sin ZP . sin P
(from which ZQ is found).

(2) Cos ZP = cos ZQ . cos QP
or cos PQ = cos PZ . sec ZQ .
(from which PQ is found).
and PS − PQ = QS,

(3) and cos ZS = cos ZQ . cos QS.

This method is shorter and more advantageous than those above, since it is more convenient in the use of the logarithms: *e.g.*, in (1) and (2), we have at sight, sin ZP and cos ZP; sin ZQ and sec ZQ: cos ZQ in (3); from the occurrence of these terms, time is saved in the computation: the same elements being prepared beforehand as in the other methods.

*Fourth Method of Computing Altitudes.**

In any spherical triangle ABC,

$$\text{Sin}\,\frac{A}{2} = \sqrt{\frac{\sin\frac{1}{2}(a+c-b)\,.\,\sin\frac{1}{2}(a+b-c)}{\sin b\,.\,\sin c}}$$

Hence, in the spherical triangle, ZPS. (See last fig.)

$$\text{Sin}\,\frac{h}{2} = \sqrt{\frac{\sin\frac{1}{2}(z+p-l')\,\sin\frac{1}{2}(z-p+l')}{\sin p\,.\,\sin l'}}$$

$$\text{or } \sin p \sin l' \,.\, \sin^2\tfrac{1}{2}h = \sin\tfrac{1}{2}(z+p-l')\,\sin\tfrac{1}{2}(z-p+l')$$

$$\text{and } \sin p \sin l' \,.\, 2\sin\frac{^2h}{2} = 2\sin\tfrac{1}{2}(z+p-l')\sin\tfrac{1}{2}(z-p+l')$$

$$= \cos(p-l') - \cos z$$
$$= -\cos z + \cos(p-l')$$
$$= 1 - \cos z - 1 + \cos(p-l')$$
$$= (1-\cos z) - (1-\cos\overline{p-l'})$$

$$\text{hence, } 1 - \cos z = 1 - \cos\overline{p-l'} + \sin p \sin l' \; 2\sin\frac{^2h}{2}$$

but $p - l' =$ mer zen dist

therefore $1 - \cos z = 1 - \cos \text{MZD} + \cos \text{dec} \cos \text{lat} \,.\, 2\sin^2\tfrac{1}{2}h$

or $\sin \text{alt} = \cos \text{MZD} - \cos \text{dec} \cos \text{lat} \,.\, 2\sin^2\tfrac{1}{2}h$

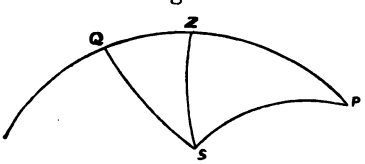
But $2\sin\frac{^2h}{2} = 1 - \cos h = \text{vers}\, h$

therefore, $\sin \text{alt} = \cos \text{MZD} - \cos \text{dec} \cos \text{lat} \,.\, \text{vers}\, h$.

Latitude by Single Altitude.

When the time is known exactly, the latitude may be found as follows:—

Fig. 43.



We have given in the triangle ZPS the true apparent time, viz., the angle P; the altitude, and consequently ZS, the zenith distance; and the polar distance PS. From S draw SQ perpendicular to PQ, then,

$$\text{Sin SQ} = \sin \text{SP} \,.\, \sin \text{P}$$
$$(1)\ \text{Sin SQ} = \sin p \,.\, \sin h$$
$$\text{and } \cos \text{PS} = \cos \text{PQ} \cos \text{QS}$$

* Dr. Mackay "on Longitude."

whence cos PS . sec QS = cos PQ

(2) Or cos p . sec QS = cos PQ

Again, cos ZS = cos ZQ cos QS

whence cos ZS . sec QS = cos ZQ

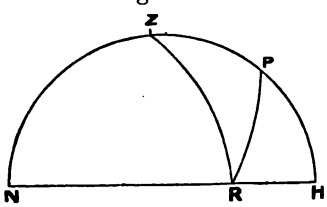
(3) Or cos z . sec QS = cos ZQ

and PQ − ZQ = Pz (the colatitude)

remembering that if PS the polar distance is greater than 90°, PQ must be subtracted from 180°, before using (3).

For Computing Time of Sunrise.

Fig. 44.



Let NH represent the horizon: NZPH the meridian: Z the zenith: P the pole. Let R be the place of the Sun's setting. Then PR is his polar distance at setting: PZ is the colatitude: and ZR = 90°, and the angle ZPR is the apparent time of sunset. Then,

$$\text{Cos ZPR} = \frac{\cos \text{ZR} - \cos \text{ZP} \cos \text{PR}}{\sin \text{ZP} \cdot \sin \text{PR}}$$

$$\text{or cos } h = \frac{\cos 90 - \cos \text{colat} \cdot \cos p.d.}{\sin \text{colat} \cdot \sin pd.}$$

But cos 90° = 0

hence cos h = − cot colat . cot $p.d.$

or cos h = − tan lat . cot $p.d$

The − sign is thus compensated for.

The lat is always under 90°: therefore tan lat is + . Then if the polar distance is more than 90°: cot p is −, and the signs of the expression are

Cos h = − tan lat . cot pd

= − . + . − = +

or cos h is + and h is acute.

If cot p is +, (*i. e.*, if the polar distance be acute) cos h is −, and h is obtuse. In the form of a practical rule: If p is greater than 90°, h is less: and *vice versâ*. To make h obtuse, it must be subtracted from 180°, for cos (180 − h) = − cos h.

Time of a Star's Rising or Setting.

The refraction of a body when on the horizon is 33′. As refraction increases an altitude, a body will appear to be on the horizon when it is really 33′ below it: or 33′ above it when really upon it. The dip also, we have seen, has the same effect, viz., to increase the altitude. Now, if we add the dip and refraction together algebraically, we have a negative altitude. Thus, if the height of the eye be 20 feet,

	′	″
Dip = −	4.	24
Ref = −	33.	0
Total −	37.	24

This quantity is called the depression.

Now, referring to the preceding figure,

$$\text{Cos } h = \frac{\cos z - \cos l' \cos p}{\sin l' \sin p}$$

$$= \frac{\text{sin alt} - \sin l \,.\, \cos p}{\cos l \sin p}$$

Subtract both sides from 1, then

$$1 - \cos h = 1 - \left(\frac{\text{sin alt} - \sin l \cos p}{\cos l \sin p}\right)$$

$$= \frac{\cos l \sin p - \sin a + \sin l \cos p}{\cos l \sin p}$$

$$= \frac{\sin (p + l) - \sin a}{\cos l \sin p}$$

But the + alt becomes − depression

therefore

$$1 - \cos h = \frac{\sin (a + l) + \sin d}{\cos l \sin p}$$

$$\text{or } 2 \sin \frac{^2h}{2} = \frac{2 \sin \tfrac{1}{2} (a + l + d) \cos \tfrac{1}{2} (a + l - d)}{\cos l \sin p}$$

$$\text{or } \sin \frac{h}{2} = \sqrt{\sin \text{S} \,.\, \cos (\text{S} - d) \,.\, \sec l \,\text{cosec}\, p}$$

And if we subtract this angle from 24^h, and to the result add the star's RA, and subtract the mean Sun's RA, we have the time of the object's rising.

Equation of Equal Altitudes.

If two observations of the Sun be taken, one on each side of the meridian, and the altitudes thus obtained be equal, it will be found that the easterly and westerly hourly angles, at the times of observation are not equal. In the interval between the A.M. and P.M. altitudes thus observed, the Sun's polar distance has been changing. Half the difference of the hour angles when the Sun had equal altitudes, therefore, is called the "equation of equal altitudes." It is defined as a correction to be applied to the middle time shown by the chronometer, resulting from the times A.M., and P.M. when the Sun had equal altitudes, to obtain the time shown by the chronometer at noon, or when the Sun was on the meridian.

If equal altitudes of a fixed star be taken, as the polar distance changes so slightly in the interval, the star will be on the meridian at the true middle time of observation. Then the difference between the middle time by chronometer, and the true time of the star's transit (found by subtracting the Mean Sun's RA from the star's RA), will be the error of the chronometer, when the object was on the meridian.

Investigation.

Referring to the triangle ZPS, as given in the "Investigation for computing the hour angle of an object," we have,

$$\text{Cos } z = \cos p \cos l' + \sin p \,.\, \sin l' \,.\, \cos h$$

An increase in the polar distance will produce a decrease in the resulting hour angle; let p become $(p + c)$, then h becomes $(h - e)$, then,

$$\begin{aligned}
\text{Cos } z &= \cos(p+c)\cos l' + \sin(p+c)\,.\,\sin l'\,.\cos(h-e) \\
&= \cos p \cos c \cos l' + \sin p\,.\,\sin c \cos l' + \left\{ \sin p \cos c \sin l' + \cos p \sin c \sin l' \right\}\,.\, \times \cos(h-e) \\
&= \cos p \cos c \cos l' + \sin p \sin c \cos l' + \left\{ \sin p \cos c \sin l' + \cos p \sin c \sin l' \right\} \times \left\{ \cos h \cos e + \sin h \sin e \right\} \\
&= \cos p \cos c \cos l' + \sin p \sin c \cos l' + \sin p \cos c \sin l' \cos h \cos e \\
&\quad + \cos p \sin c \sin l' \cos h \cos e + \sin p \cos c \sin l' \sin h \sin e \\
&\quad + \cos p \sin c \sin l' \sin h \sin e
\end{aligned}$$

But c and e being small, their cosines $= 1$ nearly.

$$= \overline{\cos p \cos l'} + \sin p \sin c \cos l' + \overline{\sin p \sin l' \cos h} + \cos p \sin c \sin l' \cos h + \sin p \,.\, \sin l' \sin h \sin e + \underline{\cos p \sin c \sin l' \sin h \sin e}$$

But $\cos p \cos l' + \sin p \sin l' \cos h = \cos z$

This destroys $\cos z$ on the left hand side, and rejecting the last term, which contains both $\sin c$ and $\sin e$, we have,

Then

$$0 = \sin p \,.\, \sin c \cos l' + \cos p \sin c \sin l' \cos h + \sin p \sin l' \,.\, \sin h \sin e$$

transposing and changing signs,

$$\sin p \sin l' \sin h \,.\, \sin e = \sin c \left\{ \sin p \cos l' - \cos p \sin l' \cos h \right\}$$

Divide by $\sin p \,.\, \sin l' \sin h$, then,

$$\sin e = \sin c \left\{ \cot l' \operatorname{cosec} h - \cot p \cot h \right\}$$

But $\sin e'' = e''$ sin $1''$ nearly

and $\sin c'' = c'' \,.\, \sin 1''$ „

then,

$$e'' = c'' \,.\, \tan \text{lat} \operatorname{cosec} h - c \tan \text{dec} \cot h$$

These parts are additive when the latitude and declination are of opposite names. Convert e'' into time. And the "equation" must be applied + to the mid time by chronometer, if the Sun's pd is increasing (*i. e.*, reckoned from the pole of observer's hemisphere), and —, if decreasing.

The Effect of an Error in Altitude upon the Hour Angle Deduced from that Altitude.

* (1) $\text{Cos } z = \cos p \cos l' + \sin p \sin l' \cos h.$

Let z become $(z+x)$

then h becomes $(h+y)$, and

$$\cos(z+x) = \cos p \cos l' + \sin p \,.\, \sin l' \cos(h+y)$$

$$\cos z \cos x - \sin z \sin x = \cos p \cos l' + \sin p \sin l' \cos h \cos y - \sin p \sin l' \,.\, \sin h \sin y$$

$\cos x$ and $\cos y$, $= 1$ nearly,

$$\cos z - \sin z \,.\, \sin x = \cos p \sin l + \sin p \cos l \cos h - \sin p \cos l \sin h \sin y$$

but $\cos z = \cos p \sin l + \sin p \cos l \cos h$ (1)*

Subtracting

$$\sin z \sin x = \sin p \cos l \sin h \sin y$$

or,

$$\frac{\sin y}{\sin x} = \frac{\sin z}{\sin p \cos l \,.\, \sin h}$$

But $\sin p : \sin z :: \sin Az : \sin h$

or $\sin p \,.\, \sin h = \sin z \,.\, \sin Az$

therefore,

$$\frac{\sin y}{\sin x} = \frac{\sin z}{\cos l \,.\, \sin z \sin Az} = \frac{1}{\cos l \,.\, \sin Az}$$

But

$$\frac{\sin y}{\sin x} = \frac{y}{x} \text{ nearly}$$

and y is the error in the hour angle

„ x „ „ „ altitude

therefore,

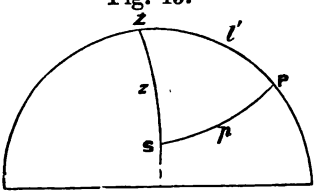
$$\frac{\text{Error in hour angle}}{\text{Error in alt}} = \frac{1}{\sin Az \,.\, \cos \text{lat}}$$

hence

Error of hour angle = (error of alt) . sec lat . cosec Az

WHEN A BODY IS OBSERVED UPON THE PRIME VERTICAL.

Fig. 45.



Let ZS be the prime vertical: then the angle PZS = 90°: S the position of an object; then by Napier's rules,

Cos PS = cos PZ . cos ZS

or $\cos p = \cos l' \,.\, \cos z$

whence (1) sin dec = sin lat . sin alt

Again, $\sin z = \sin p \,.\, \sin P$

(2) or cos alt = cos dec sin hour angle

also, $\sin PZ = \cot P \,.\, \tan ZS$

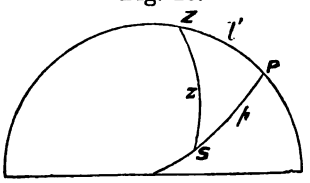
(3) or cos lat = cot hour angle . cot alt.

and $\cos P = \cot p \,.\, \tan l'$

(4) or cos hour angle = tan dec cot lat .

When the Object is on the Six O'clock Hour Circle.

Fig. 46.



In this case the right angle is at P: and by Napier's rules we have the following,—

Cos z = cos l' cos p

(1) or sin alt = sin lat sin dec

and cos Z = cot z . tan l'

(2) or cos azimuth = tan alt cot lat

sin p = sin z . sin Z

or cos dec = cos alt . sin azimuth

When an Object is on the Horizon.

(See fig. for Investigation of Amplitude).

We have the angle ZPE = time of sunset

PB = latitude

PE = polar dist

then, cos EPB = cot PE . tan PB

(1) — cos hour angle = tan dec tan lat

again, sin PB = cot EPB . tan EB

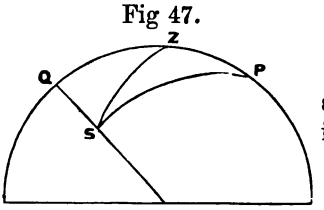
sin lat = — cot hour angle . tan EB

= — cot hour angle . cot EW

(2) or sin lat = — cot hour angle cot amp

When a Body is on the Equator.

Fig 47.



Here PQS is the right angle. Suppose the altitude and hour angle to be known: to find the latitude, we have,

QS = angle QPS

for P is the pole of QS; and

Cos ZS = cos ZQ cos QS

or sin alt = cos lat . cos hour angle

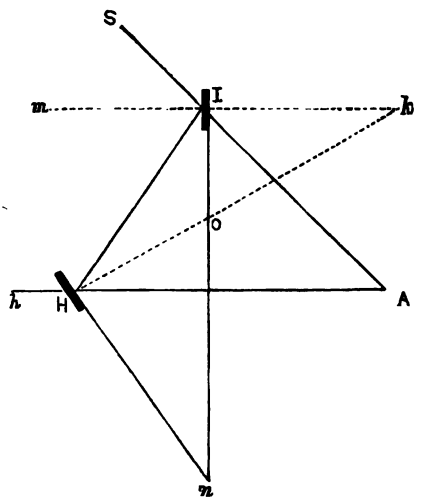
whence, cos lat = sin alt . sec hour angle.

Principle of the Sextant.

The sextant is an instrument used for the purpose of measuring angles, and is moveable in any plane, so that the angular distance between two objects may be observed. The arc contains the sixth part of a circle, hence its name, but as the arc is graduated doubly fine, the instrument enables us to measure angles as far as 120°.

A similar but rougher instrument divided less minutely and measuring angles up to 90°, is called a quadrant. The principle of both instruments is the same.

Fig. 48.



Let H and I represent two mirrors standing perpendicular to the surface of this page. S the position of any heavenly body. A that of an observer's eye. A*h* his horizon. Then the angle SAH will measure the altitude of S, for it will be the same as the arc from S to *h* of a circle of altitude of which A is the centre. Join HI, make the angle SI*m* equal to the angle *m*IH. Produce *m*I, and draw HK perpendicular to the surface of the mirror H, and meeting *m*I produced in K. Let the lines of direction of the mirrors I and H be produced to meet each other in *n*.

Then the angle I*n*H measures the inclination of the two mirrors I and H. We have first to prove that the inclination of the mirrors measures half the altitude of the Sun, or that

$$n = \tfrac{1}{2}\text{A}.$$

In the triangles I*o*K and H*on*.

The angle I*o*K = H*on* (Euc. I. 15)

and *o*IK = *o*H*n* (each 90°)

therefore K = *n*.

Again, in the triangle IHA,

SIH = IHA + A (Euc. I. 32)

or $\tfrac{1}{2}$ SIH = $\tfrac{1}{2}$IHA + $\tfrac{1}{2}$A

i.e. *m*IH = IHK + $\tfrac{1}{2}$A

But in the triangle IHK,

*m*IH = IHK + K (Euc. 32)

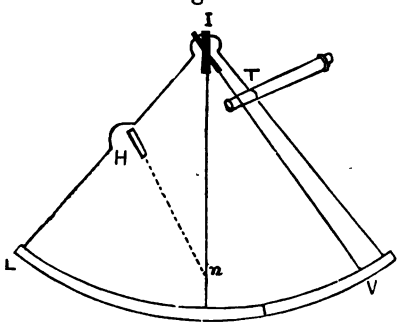
hence,

IHK + $\tfrac{1}{2}$A = IHK + K

omit IHK
then $\frac{1}{2}A = K$
But $K = n$
therefore $n = \frac{1}{2}A$

The mirror I is entirely silvered: H is only half so. Supposing therefore an artificial horizon filled with liquid to be in use, the reflected image of S would be seen through the unsilvered part of the mirror H, while the true image of S would be seen in the mirror I, and be reflected in the silvered part of H, and two images of S would be seen through H.

Fig. 49.



The glasses are attached to a framework, and to I is fastened a moveable radius IV, so that as the extremity V is moved along the limb LV the inclination of the glasses I and H is altered. A graduated scale called the Vernier, (from its inventor, Pierre Vernier, 1631), is attached to IV, and upon it are marked sub-divisions of the limb LV.

In good instruments each degree is divided into six equal portions (10′ each). The divisions of the Vernier are shorter; for if each division on the limb be 10′, then 59 such divisions would be 590′ or 9′.50″, this constitutes the Vernier scale, and the value of each division upon it would be 10″ shorter than the corresponding one on the limb. Thus six divisions of the Vernier are just 1′ shorter than six of the limb; and if the six divisions of the Vernier were to coincide with any stroke of the limb, the zero point of the former would be just 1′ in advance of the corresponding stroke on the limb. The ten minute divisions are divided into single minutes which are distinguished by longer strokes on the Vernier to facilitate the counting.

Let n represent the value of a division on the limb*
m on the Vernier

* Riddle's Navigation, pages 89, 90.

then mn = value of these in divisions, and this being divided into $m + 1$ equal parts give $\frac{mn}{m+1}$ for each division of the Vernier; therefore $n - \frac{mn}{m+1} = \frac{n}{m+1}$ the difference between one division of the limb, and one of the Vernier.

But in the sextant $n = 10'$, and if we wish to divide into $10''$, we have this equation,

$$\frac{n}{m+1} = 10'' \text{ or } \frac{10'}{m+1} = 10'', \text{ or } \frac{600''}{m+1} = 10''$$

whence $m = 59$

119 divisions of the limb are thus taken for the length of the Vernier.

Adjustments.

The adjustments are as follow:

I.—To make the index glass I perpendicular to the plane of the instrument.

Turn the sextant with its face upwards and move the Vernier scale near to the centre of the limb; then the half of the limb reflected in the glass should be contiguous to the half which is not reflected in it; but if the reflected part appears lower than the true, tighten the adjusting screw and *vice versâ*.

II.—To make the horizon glass perpendicular to the plane of the instrument.

Screw in the telescope, and look directly through it at a heavenly body, then if the reflected image does not pass over the direct image upon moving the Vernier, the adjustment can be made by placing the pin, to be found in most sextant cases, through the screw in the head of the horizon glass H, and turning it till the images pass exactly over each other.

III.—To make the two glasses parallel to each other.

Move the Vernier to zero so that O on the Vernier shall coincide with O on the limb. Look through the horizon glass, and then, if the edge of the horizon as seen through the unsilvered part, be contiguous to that seen through the silvered portion, the instrument is in adjustment; if not, turn the lower screw at the back of the glass until the adjustment is complete.

IV.—To make the axis of the telescope parallel to the plane of the instrument.

Make one of the wires in the focus of the telescope parallel to the plane of the instrument. Select two heavenly bodies not less than 90° distant from each other, bring them together on this wire; then by moving the instrument slightly, the two bodies can be brought to the other wire where they should be also in contact. If they are not, the adjustment can be made by moving a screw in the collar farthest from the instrument. This is called "collimation error," and when once rectified is not liable to change.

Care in using a Sextant.

The following valuable information respecting the care that should be taken in handling this instrument, is extracted from "Pryde's Navigation," page 369, and it should be specially noted by learners:—

1. When the brilliancy of two objects under observation is very unequal the telescope should be brought opposite the unsilvered part of the horizon glass by the large screw attached to the telescope collar, as the brilliancy is by this means rendered more nearly equal.

2. In testing the adjustments, the great heat of the sun will affect the sextant, if it be too long exposed.

3. The various powers of the telescope should be practised with, as a good observation often depends upon using a suitable power.

4. In observing an angle between two objects in motion, the best contact is obtained by setting the sextant a little in advance, and waiting for the contact.

5. In taking lunar distances it is advantageous to set the Vernier a little in advance of the distance, screw in the telescope and await the contact.

6. The screws, tubes, etc., should be turned, pushed in and out as little as possible. The sextant should be held by the handle when in use, and very carefully when taken out and put into its case, be held by its centre; any rough usage destroys all adjustment and very soon tells upon the value of the instrument.

Index Error.

When a sextant is in adjustment and the index and horizon glasses are parallel, the zero-point or arrow-head on the Vernier

should exactly coincide with 0° on the limb. These glasses are parallel, and in adjustment, when by pointing at any well-defined object, (the Sun for example), the reflected image from the index glass and the direct image as seen through the horizon glass, coincide with each other. If when the mirrors are parallel, it be found that zero on the Vernier falls to the right or left of 0° on the limb, its distance on the arc from 0° is called the **index error.**

If zero falls to the left hand of 0°, or on the arc, then every altitude measured by the sextant will be too great, and the index error is —. If, on the contrary, the index fall to the right of 0°, every arc read off will be too little, and the index error is +.

To Determine the Index Error.

1. Set the Vernier nearly at 0, and fasten it by the "clamp screw." Then, pointing to the Sun, turn the "tangent screw," until the reflected image coincides with the direct image. The arc read off from the limb is the index error + or —.

2. By pointing at the Sun as before, get a series of contacts between the limbs of the reflected and direct image on each side. The readings of the contacts on one limb will be to the left of 0°, or on the arc, and of the other, to the right of 0°. The latter readings (off the arc) must be taken from 10 if the sextant be divided into 10′ divisions; or, from 15′ if divided into 15′. A quarter of the sum of the two "means" of the readings will be the Sun's semi-diameter for the day; and half their difference the index error.

Example.

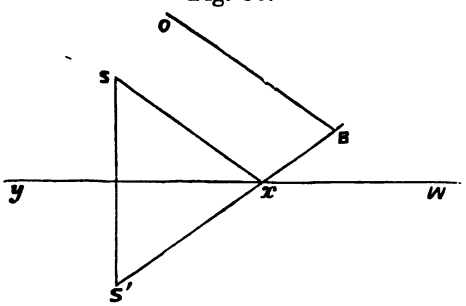
Readings (Left) On the arc.				Readings (Right) Off the arc.			
	30′	45″			32′	30″	
	30	50			32	45	
	31	0			32	50	
3)	92	35		3)	98	5	
L	30	51	6	R	32—41		6
R	32	41	6	L	30	51	6
4)	63	33	2	2)	1	50	0
Sun's SD	15	53	3	I.E.	+ 0	55	

+ because R is greater than L

The Artificial Horizon.

The artificial horizon is a trough filled with liquid, in which when the sensible horizon is obscured, an altitude of a celestial body may be taken. Its use is confined in nearly every case to observations on shore. The following figure proves that the altitude taken in an artificial horizon, is double that measured above the sensible horizon.

Fig. 50.



Let B be the position of an observer's eye, then if S be a celestial object, the angle S*xy* is its altitude, and the image of S, viz., S′ will appear to be as far below the surface of the liquid *yx*, as S is above it: or according to the principle laid down in optics, "The angle of incidence S*xy* at which a ray of light S*x* strikes a surface, is equal to the angle at which the body is reflected, viz., S′*xy*." Join BS′, meeting *yw* in *x*. Join S*x*. Then as S is at an immense distance from B compared with *x*, S*x* is parallel to BO. The angle OBS′ is the angle measured with the sextant, in the artificial horizon *yx*.

$$\text{The angle OBS}' = 180 - \text{S}x\text{B}$$
$$\text{But S}xy + \text{B}xw = 180 - \text{S}x\text{B}$$
$$\text{therefore, OBS}' = \text{S}xy + \text{B}xw$$

But S*xy* and B*xw*, are equal,

$$\text{therefore OBS}' = 2\ \text{S}xy$$

or the angle measured is double the altitude of the object. The advantage in this is, that any error made, is halved in obtaining the altitude. "Merrifield's Navigation," p. 152.

EXAMPLES FOR PRACTICE,

IN

PLANE AND SPHERICAL TRIGONOMETRY.

EXAMPLES FOR PRACTICE,

IN

PLANE AND SPHERICAL TRIGONOMETRY.

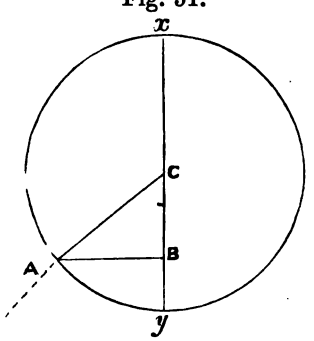
Plane Right Angled Triangles.

1. In the right angled plane triangle ABC, given, B = 90°.

Given, AC = 230.5 : ∠A 53° 20′ 10″ : find the other parts.

(1) *By Gunter's Scale.*

Fig. 51.



Draw a vertical line *xy*. On the scale of chords extend the compass leg to 60°. Take a point C in the line *xy* as a centre, and with this chord 60° as a radius, describe a circle of *y*A*x*.

From *y* to A lay off *y* A = from the scale of chords = 36° 40′ (∠C, the complement of A). From the scale of equal parts take CA = 230. Then from A draw AB perpendicular to C*y*. Then it will be found that on the same scale from which AC is taken, AB measures 137, and BC 185 nearly.

(2) *By Inspection.*

Enter the inspection table with angle C as course 37°, and AC as dist 230; where under the dep column AB = 138·4 and BC (the *d* lat column) = 183·7.

(3) *By Computation.*

By referring to the formulæ for the solution of right angled triangles, since AC and A are given, we have,

$$BC = AC \cdot \sin A, \text{ and } AB = AC \cdot \text{Cos } A .$$

whence

$$\log BC = \log AC + \log \sin A - 10 \text{ and } \log AB = \log AC + \log \cos A - 10$$

$\log AC = 2{\cdot}362671$	$\log AC = 2.362671$
$\log \sin A = 9{\cdot}904256$	$\log \cos A = 9{\cdot}776062$
$BC = 184{\cdot}8 \ldots 2{\cdot}266927$	$AB = 187{\cdot}6 \ldots 2{\cdot}1387333$

In like manner, by using the other formulæ which bring in the two given parts, all cases of right angled plane triangles may be solved.

Oblique Angled Plane Triangles.

CASE I.

Where two sides and an angle opposite one of them are given, or two angles and a side opposite or adjacent to them.

1. In an oblique angled plane triangle ABC, given $AC = 23{\cdot}01$; $AB = 19{\cdot}42$: $\angle B = 39° 42' 15''$, find the other parts.

By referring to (Class VII. 1) Plane Trigonometry, it will be seen that in any plane triangle, "the sides are to each other as the sines of their opposite angles," or that,

$$a : b :: \sin A : \sin B$$

In practice, the most convenient way of using this rule is:

$$a : \sin A :: b : \sin B$$

If in any example in this case two sides and one angle are given, begin with a side; thus, a, b, and A are given to find B,

$$a : \sin A :: b : \sin B$$

$$\text{or } \sin B = \frac{b \cdot \sin A}{a} .$$

Suppose A, B, and a are given, then,

$$\sin A : a :: \sin B : b$$

$$\text{or } b = \frac{a \cdot \sin B}{\sin A} = a \sin B \operatorname{cosec} A$$

In the examples before us, AC, AB, and B are given. AC is opposite to B; hence

$$AC : \sin B :: AB : \sin C$$

$$\text{and } AC \cdot \sin C = AB \cdot \sin B$$

$$\text{or } \sin C = \frac{AB \cdot \sin B}{AC}$$

$$\begin{aligned} \log AB &= 1{\cdot}288249 \\ \log \sin B &= 9{\cdot}805381 \\ & \underline{\quad\quad} \\ & 11{\cdot}093630 \\ \log AC &= 1{\cdot}361917 \\ & \underline{\quad\quad} \\ C = 32^\circ\ 37'\ 21'' \sin & \ 9{\cdot}731713 \end{aligned}$$

But since the three angles $A + B + C = 180^\circ$

$$\begin{aligned} \text{and } B &= 39.\ 42.\ 15 \\ \text{,, } C &= 32.\ 37.\ 21 \\ & \underline{\quad\quad\quad} \\ & 72.\ 19.\ 36 \\ & 180.\ 0.\ 0 \\ \text{therefore } A &= 107.\ 40.\ 24 \end{aligned}$$

To compute the remaining side, we have three angles and two sides. Begin with an angle, thus:

$$\text{Sin B} : AC :: \sin A : BC$$

$$\text{or } BC = AC \cdot \sin A \cdot \operatorname{cosec} B.$$

$$\begin{aligned} \log AC &= 1{\cdot}361917 \\ \log \sin A &= 9{\cdot}979003 \\ \log \operatorname{cosec} B &= 10{\cdot}194619 \\ & \underline{\quad\quad} \\ BC\ 34{\cdot}32 &= 1{\cdot}535539 \end{aligned}$$

CASE II.

Where two sides and the included angle are given.

The rule for the solution of this case is to be found in (Class VII. 2), Plane Trigonometry. When two sides, a, b, and angle C are given,

$$(a + b) : (a - b) :: \tan \tfrac{1}{2}(A + B) \cdot \tan \tfrac{1}{2}(A - B).$$

$\frac{1}{2}(A + B)$ is found by subtracting angle C from 180°, and dividing by 2; and when this is obtained,

$$\text{Tan } \tfrac{1}{2}(A - B) = \frac{(a - b) \cdot \tan \tfrac{1}{2}(A + B)}{(a + b)}$$

$$\text{whence } \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A$$

$$\text{and } \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B$$

and the solution then depends upon the last case, with two angles and an opposite side.

Example: Given $a = 29{\cdot}85 : b = 21{\cdot}65 :$ C 101° 31′ 25″; to find the other parts.

$$(a + b) : (a - b) :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B)$$
$$\text{or } 51{\cdot}5 : 8{\cdot}2 :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B)$$

$$\begin{array}{rr} & {}^{\circ}\ \ {}'\ \ {}'' \\ C = & 101.\ 31.\ 25 \\ & 180.\ \ 0.\ \ 0 \\ & \overline{2)78.\ 28.\ 35} \\ \tfrac{1}{2}(A + B) & 39.\ 14.\ 17 \end{array}$$

$$51{\cdot}5 : 8{\cdot}2 :: \tan 39° 14′ 17″ : \tan \tfrac{1}{2}(A - B)$$
$$\text{whence, } \tan \tfrac{1}{2}(A - B) = \frac{8{\cdot}2 \times \tan 39° 14′ 17″}{51.5}$$

$$\begin{array}{rr} \log 8{\cdot}2 = & 0{\cdot}913814 \\ \log \tan 39° 14′ 17″ = & 9{\cdot}912055 \\ & \overline{10{\cdot}825869} \\ \log 51{\cdot}5 = & 1{\cdot}711807 \\ \tfrac{1}{2}(A - B) = 8° 0′ 34″ \tan & \overline{9{\cdot}114062} \end{array}$$

$$\begin{array}{rr} \tfrac{1}{2}(A + B) = & 39\ 14\ 17 \\ A & 47\ 14\ 51 \\ B & 31\ 13\ 43 \end{array}$$

To find AB

$$\text{Sin A} : \text{BC} :: \sin \text{C} : \text{AB}$$
$$\text{AB} = \text{BC} \cdot \sin \text{C} \operatorname{cosec} \text{A}$$

$$\begin{array}{rr} \log \text{BC} = & 1{\cdot}474944 \\ \log \sin \text{C} = & 9{\cdot}991156 \\ \log \operatorname{cosec} \text{A} = & 10{\cdot}134131 \\ \text{AB } 39{\cdot}83 = & 1{\cdot}600231 \end{array}$$

CASE III.

Three sides given to find an angle. By (Class VII. 4), where a, b, c, are given.

$$\operatorname{Cos} \tfrac{1}{2} A = \sqrt{\frac{S \cdot (S - a)}{bc}}$$

Given AC = 24·87 : AB = 25·67 : BC = 31·25 ; find angle A.

a = 31·25			
b = 24·87			
c = 25·67			
2)81·79			
S 40·89	log 1·611617		log b = 1·395676
a 31·25			log c = 1·409426
(S − a) 9·64	log 0·984077		bc = 2·805102
S (S − a) + 20	22·595694		
bc . . .	2·805102		
° ′ ″	2)19·790592		
½ A 38. 12. 28	cos 9·895296		
A 76. 24. 56			

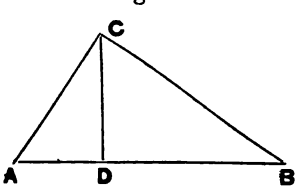
Another Method.

The following method, though little used, will find an angle, when three sides are given.

As the base, or longest side, is to the sum of the other two sides, so is the difference of those sides, to the difference of the segments of the base, made by a perpendicular drawn from the opposite angle. Then the half difference added to half the base, gives the segment adjacent to the greater side, and subtracted, gives the segment adjacent to the less side.

Example:—Given AB = 30 : AC = 21 : BC = 23.

Fig. 52.



As AB : (AC + CB) :: (AC − CB) : (AD − DB)

or 30 : 44 :: 2 : (AD − DB)

whence (AD − DB) = $\frac{88}{30}$ = 2 . 93

or by logs, log 88 = 1·944483
log 30 = 1·477121
(AD − DB) 2·93 = 0·467362
½(AD − DB) 1·46
½ AB 15·00.
BD 16·46
AD 13·54

In the right angled triangle DAC,

$$\frac{AD}{AC} = \cos A$$

log AD + 10 = 11·131619
log AC = 1·322219
A = 49° 51′ 9″ cos 9·809400

Another Method for Case II.

The following may be advantageously employed in the computation of the third side direct, when two sides and the included angle are given.

Given b, c, and A; to find a.

We know that,

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2\,bc}$$

$$\text{and } 2\,bc \cos A = b^2 + c^2 - a^2$$

$$\text{or } a^2 = b^2 + c^2 - 2\,bc \cos A$$

$$\text{but } \cos A = 2 \cos^2 \tfrac{1}{2} A - 1$$

therefore,

$$a^2 = b^2 + c^2 - 2\,bc\,(2 \cos^2 \tfrac{1}{2} A - 1)$$
$$= b^2 + c^2 + 2\,bc - 4\,bc \cos^2 \tfrac{1}{2} A$$
$$= (b + c)^2 - 4\,bc \cos^2 \tfrac{1}{2} A$$

$$\text{Let } 4\,bc \cos^2 \tfrac{1}{2} A = M^2$$

$$\text{then } a^2 = (b + c)^2 - M^2$$

$$\text{or } a = \sqrt{(b + c + M) \,.\, (b + c - M)}$$

Example:

Given $b = 216{\cdot}8$: $c = 314{\cdot}7$: A = 112° 13′

$$M = \sqrt{4\,bc \,.\, \cos^2 \tfrac{1}{2} A}$$

$$\text{and } a = \sqrt{(b + c + M)\,(b + c - M)}$$

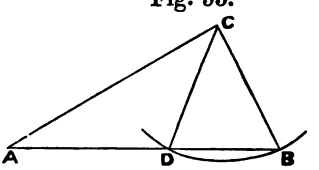
$\frac{1}{2}$ A = 56° 36′ 30″	cos	9·740647
		2
		19·481294
	log 4 =	0·602060
b 216·8 . .	log b =	2·336059
c 314·7	log c =	2·497897
(b + c) 531·5		2)4·917310
M 287·5		2·458655
(b + c + M) 819·0	log . .	2·913284
(b + c − M) 244·0	log . .	2·387390
		2)5·300674
	a = 447 =	2·650337

AMBIGUOUS CASE OF PLANE TRIANGLES.

When two sides are given, and the angle opposite the less side, each of the unknown parts will have two distinct values. In the triangle ABC, if BC and AC are given, and angle A, from C as a centre, and radius CB, describe the arc BD, and join CD. Each of the triangles ABC, and ADC, has AC, CB, and angle A belonging to it.

Example: In the triangle ABC, given, BC = 165 : AC = 232 and angle A 21° 25′ to find the other parts.

Fig. 53.



To find angle B

BC : sin A :: AC : sin B

$$\text{or } \sin B = \frac{AC \cdot \sin A}{BC}$$

	log AC =	2·365488
	sin A =	9·562468
		11·927956
	BC =	2·217484
	B 30° 53′ = sin	9·710472

	° ′
1st Solution B =	30. 53
	180. 0
2nd Solution B =	149. 7

1st Solution.

	° ′
B =	30. 53
A =	21. 25
	52. 18
	180. 0
C =	127. 42

2nd Solution.

	° ′
B =	149. 7
A =	21. 25
	170. 32
	180. 0
C =	9. 28

To find AB.

Sin A : BC :: sin C : AB	Sin A : BC :: sin C : AB
AB = BC . sin C . cosec A	AB = BC . sin C . cosec A
log BC = 2·217484	log BC = 2·217484
log sin C = 9·898299	log sin C = 9·216097
log cosec A = 10·437532	log cosec A = 10·437532
AB = 357·5 2·553315	AB = 74·32 1·871113

To find the Area.

By (Class VII. 8),

$$\text{Area} = \sqrt{S \cdot (S - a)(S - b)(S - c)}$$

Example: Given $a = 30 : b = 45 : c = 50$: to find the area.

$a =$	30
$b =$	45
$c =$	50
	2)125
S	62·5 log 1·795880
(S − a)	32·5 log 1·511883
(S − b)	17·5 log 1·243038
(S − c)	12·5 log 1·096910
	2)5·647711
Area 666·6 =	2·823855

The other expressions adapted to logarithms are,

When two sides and included angle are known,

$$\text{Area} = \tfrac{1}{2}\, ab \,.\, \sin C$$

And when the base and perpendicular are known,

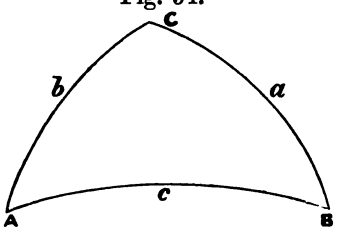
$$\text{Area} = \tfrac{1}{2} \text{ base} \times \text{perpendicular.}$$

Spherical Triangles.

CASE I.

Two sides and the contained angle given, to find the third side, direct.

Fig. 54.



In the oblique angled triangle ABC, given

	°	′	″
AC =	120.	15.	30
CB =	110.	45.	15
C =	45.	20.	30

to find AB.

By Formula 10, *in Spherical Trigonometry.*

$$\text{Sin}^2\,\theta = \sin AC \,.\, \sin CB \,.\, \cos^2 \tfrac{1}{2} C$$

$$\text{Sin}^2 \frac{AB}{2} = \sin \left\{ \frac{AC + CB}{2} + \theta \right\} \sin \left\{ \frac{AC + CB}{2} - \theta \right\}$$

C	$= 45.\ 20.\ 30$		
$\frac{1}{2} C$	$= 22.\ 40.\ 15$	cos	9·965077
			2
			19·930154
AC	120. 15. 30	sin	9·936394
CB	110. 45. 15	sin	9·970862
	2)231. 0. 45		2)19·837410
$\frac{1}{2}(AC + CB)$	115. 30. 22	sin	9·918705
θ . . .	56. 1. 32		
$\frac{1}{2}(AC + CB) + \theta$	171. 31. 54	sin	9·168093
$\frac{1}{2}(AC + CB) - \theta$	59. 28. 50	sin	9·935233
			2)19·103326
	20. 51. 55	sin	9·551663
	2		
AB	$= 41.\ 43.\ 50$		

CASE II.

Two sides and the contained angle given, to find the other two angles.

In the spherical triangle ACB, given $a = 41°\ 30'$: $b = 56°\ 16'$: $C = 39°\ 30'$: find the angles A and B.

By Napier's First and Second Analogies.

$$\text{Tan}\,\tfrac{1}{2}(A + B) = \cos \tfrac{1}{2}(a - b) \,.\, \sec \tfrac{1}{2}(a + b) \,.\, \cot \tfrac{1}{2} C$$
$$\tan \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2}(a - b) \,.\, \text{cosec}\, \tfrac{1}{2}(a + b) \,.\, \cot \tfrac{1}{2} C$$

C	$= 39°\ 30'$				
$\frac{1}{2} C$	= 19 45	cot	10·444861	cot	10·444861
b	56 16				
a	41 30				
	2)14 46				
$\frac{1}{2}(b - a)$	7 23	cos	9·996384	sin	9·108927
	2)97 46				
$\frac{1}{2}(b + a)$	48 53	sec	10·182042	cosec	10·122990
$\frac{1}{2}(B + A)$	76 36	tan	10·623287	tan	9·676778
$\frac{1}{2}(B - A)$	25 24				
B	102 0	Because b is greater than a			
A	51 12				

CASE III.

Three sides given, to find an angle. Given $a = 12° 10'$: $b = 15° 30'$: $c = 17° 30'$: find A.

By Formula 2, Spherical Trigonometry.

$$\text{Cos } \tfrac{1}{2} \text{ A} = \sqrt{\text{Sin S} \cdot \sin (\text{S} - a) \operatorname{cosec} b \cdot \operatorname{cosec} c}$$

	° ′		
a	12. 10		
b	15. 20	cosec	10·573101
c	17. 30	cosec	10·525885
	2)45. 0		
S	22. 30	sin	9·582840
a	12. 10		
(S − a)	10. 20	sin	9·253761
			2)19·935587
½ A =	21. 47	cos	9·967793
A =	43. 34		

CASE IV.

Two angles and an included side given, to find the other two sides given A = 22° 40′ : B 31° 50′ : C = 21° 40′ : find a and b.

By Napier's Third and Fourth Analogies.

$$\text{Tan } \tfrac{1}{2} (a + b) = \text{Cos } \tfrac{1}{2} (\text{A} - \text{B}) \cdot \text{Sec } \tfrac{1}{2} (\text{A} + \text{B}) \cdot \tan \tfrac{1}{2} c$$
$$\text{Tan } \tfrac{1}{2} (a - b) = \text{Sin } \tfrac{1}{2} (\text{A} - \text{B}) \text{ Cosec } \tfrac{1}{2} (\text{A} + \text{B}) \cdot \tan \tfrac{1}{2} c$$

	° ′				
c =	21. 40				
½ c =	10. 50	tan	9·281858	tan	9·281858
B	31. 50				
A	22. 40				
	2) 9. 10				
½ (B − A)	4. 35	cos	9·998609	sin	8·902596
	2)54. 30				
½ (B + A)	27. 15	sec	10·051090	cosec	10·339254
½ (b + a)	12. 6	tan	9·331557	tan	8·528708
½ (b − a)	1. 54				
b	14. 0	because B is greater than A			
a	10. 12				

CASE V.

Two sides and an opposite angle given, or *vice versâ*

By Formula 6, *Spherical Trigonometry.*

$$\text{Sin } a : \sin b :: \sin A : \sin B$$

given $a = 72° 10' : b = 31° 15' : A = 62° 40'$

to find B

By the above formula,

$$\text{Sin B} = \frac{\sin b \,.\, \sin A}{\sin a} = \sin b \,.\, \sin A \,.\, \text{cosec } a$$

log sin b =	9·714978
log sin A =	9·948584
log cosec a =	10·021385
B 28° 56′ sin	9·684947

CASE VI.

When three angles are given, to find a side.

This is a case peculiar to spherical triangles only; for in a plane triangle the size of an angle has no effect upon the lengths of the sides which contain it, and the three angles together can never exceed 180°.

Example: Given A 59° 2′ : B 72° 54′ : C 71° 18′ : to find a

By Formula 13, *Spherical Trigonometry.*

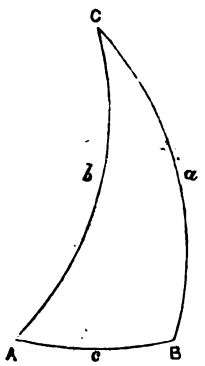
$$\text{Cos } \frac{a}{2} = \sqrt{\cos (\text{M} - \text{B}) \cos (\text{M} - \text{C}) \text{ cosec B cosec C}}$$

	° ′	
A	59. 2	
B	72. 54	cosec 10·019636
C	71. 18	cosec 10·023554
	2)203. 14	
M	101. 37	
(M − B)	28. 43	cos 9·943003
(M − C)	30. 19	cos 9·936136
		2)19·922329
$\frac{a}{2}$ =	23. 52	cos 9·961164
a =	47. 44	

CASS VII.

Right Angled Spherical Triangles.

Fig 55.



In the solution of cases in right angled spherical triangles, the following formulæ are called into use, as proved on pages 48-49, B being 90°.

for (b).

(1) $\cos b = \cos a \,.\, \cos c$

(2) $\cos b = \cot \mathrm{A} \,.\, \cot \mathrm{C}$

for A .

(3) $\cos \mathrm{A} = \cos a \,.\, \sin \mathrm{C}$

(4) $\cos \mathrm{A} = \cot b \,.\, \tan c$

for c .

(5) $\sin c = \sin b \,.\, \sin \mathrm{C}$

(6) $\sin c = \cot \mathrm{A} \,.\, \tan a$

for a .

(7) $\sin a = \sin b \,.\, \sin \mathrm{A}$.

(8) $\sin a = \cot \mathrm{C} \,.\, \tan \mathrm{c}$

for C

(9) $\cos \mathrm{C} = \cos c \,.\, \sin \mathrm{A}$

(10) $\cos \mathrm{C} = \cot b \,.\, \tan a$

By means of these formulæ any case of right angled spherical triangles may be solved.

It should be remembered in working out this case of spherical triangles, that it does not follow that we should have the unknown term on the left hand side of the equation, as in right angled plane triangles.

One of the ten formulæ above enumerated, must be chosen, which will bring in with it the two known parts, and the unknown part.

It should also be noted that each part in the computation ought to be separate and independent of any computed part going before; as it is obvious that if an error should be made in the computation of any one part, all the rest of the work would be vitiated as involving that error. The above formulæ will enable us to make each part independent in itself of every thing but the part given.

The following facts require to be committed to memory before entering upon the solutions :—

I.—The sides about the right angle are of the same affection as their opposite angles; that is to say, if A is obtuse BC is obtuse, if acute, acute. The same with AB and C.

II.—The affection of the hypothenuse is determined generally from this equation:—

$$\cos b = \cos a \,.\, \cos c$$

where the affections of two terms being known, that of the third may be determined. For by reference to the circle (Plane Trigonometry) page 2, we find, that, the sines from 0 to 180 are +, the cosines from 0 to 90 are +, those from 90 to 180 are —. Therefore in the above equation should any part be — , it must be taken from 180° when computed. The quick use of the formulæ according to the above directions, can only be acquired by good practice.

Example 1: In the right angled spherical triangle ABC, given AB = 100° 0′ : BC = 60° 10′ : to find the other parts.

(1) To find AC

Cos AC = cos AB . cos BC
(−) (−) (+)

cos AB = 9·239670
cos BC = 9·969775
85° 2′ cos 8·936445
180 0

AC 94 58 ∵ cos AC is —

(2) To find A

Sin AB = cot A . tan BC

sin AB . cot BC = cot A
sin AB = 9·993351
cot BC = 9·758517
A 60° 32′ cot 9·751868

acute, because BC is acute.

(3) To find C

Sin BC = cot C . tan AB
sin BC . cot AB = cot C
sin BC = 9·938258
cot AB = 9·246319
cot 9·184577
81° 18′
180 0

C 98 42 obtuse, because AB is obtuse.

Example 2: Given AC = 100° 42′ : A = 78° 10′

(1) To find BC.

Sin BC = sin AC . sin A

sin AC = 9·992382

sin A = 9·990671

74° 6′ sin 9·983053

acute, because A is acute.

(2) To find C

Cos AC = cot A . cot C

cos AC . tan A = cot C

cos AC = 9·268734

tan A = 10·678778

48° 27′ cot 9·947512

180

131 33 (see argument).

(3) To find AB

cos A = cot AC . tan AB

cos A . tan AC = tan AB

cos A = 9·311983

tan AC = 10·723649

47° 20′ tan 10·035632

180 0

AB = 132 40 (see argument).

Arguments for C *and* AB.

Cos AC = cos AB cos BC

since cos AC is − : and cos BC is + :

therefore cos AB is − ,

and AB is obtuse,

and since AB is obtuse,

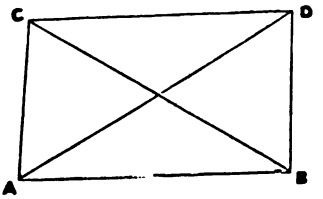
C is also obtuse.

Application of the Formulæ of Plane and Spherical Trigonometry, to the Solution of Problems in Surveying and Astronomy.

(1) *Questions involving the Solutions of Plane Triangles.*

Question I.—Being desirous of finding the distances between two distant objects D and C, I measured a base AB of 384 yards, on the same horizontal plane with the objects. At A I found the angle DAB 48° 12′, and CAB 89° 18′; at B, the angle ABC was 46° 14′, and ABD 87° 4′. It is required from these data to compute the distance between C and D.

Fig. 56.



Here we have given,

AB = 384 yards
angle CAB = 89° 18′
„ DAB = 48 12
„ DBA = 87 4
„ CBA = 46 14
to find CD.

(1) In the triangle CAB, we have CAB = 89° 18′, CBA = 46° 14′, and AB = 384, from which CB may be found.

ACB = 180° − (CAB + CBA) = 180° − (89° 18′ + 46° 14′)
= 44° 28′

Then, As, sin ACB : AB :: sin CAB : BC
whence BC = AB . sin CAB . cosec ACB

log BA =	2·584331
log sin CAB =	9·999968
log cosec ACB =	10·154595
BC 548·1 =	2·738894

(2) In the triangle DAB, we have DBA = 87° 4′, DAB = 48° 12′, and AB = 384, from which DB may be found.

ADB = 180° − (DBA + DAB) = 180° − (87° 4′ + 48° 12′)
= 44° 44′

Then, As, sin ADB : AB :: sin DAB : DB
whence DB = AB . sin DAB . cosec ADB

log AB =	2·584341
log sin DAB =	9·872434
log cosec ADB =	10·152546
BD 406·7 =	2·609311

(3) In the triangle CBD, we have CB = 548·1, BD = 406·7 and the contained angle CBD = the difference between DBA and CBA = (87° 4′ − 46° 14′) = 40° 50′, to find angles BDC and BCD.

	° ′
CBD =	40. 50
	180. 0
	2)139. 10
½ (C + D) =	69. 35

Then, As, $(CB+BD) : (CB-BD) :: \tan \frac{1}{2}(D+C) : \tan \frac{1}{2}(D-C)$

or $\tan \frac{1}{2}(D-C) = \dfrac{(CB-BD) \cdot \tan \frac{1}{2}(D+C)}{(CB+BD)}$

$$= \frac{(548{\cdot}1 - 406{\cdot}7) \cdot \tan 69^\circ 35'}{(548{\cdot}1 + 406{\cdot}7)}$$

$$= \frac{141{\cdot}4 \cdot \tan 69^\circ 35'}{954{\cdot}8}$$

$\log 141{\cdot}4 =$	2·150449
$\tan 69^\circ 35' =$	10·429191
	12·579640
$\log 954{\cdot}8 =$	2·979912
$\frac{1}{2}(D-C)\ 21^\circ 41' =$	9·599728
$\frac{1}{2}(D+C)$	69 35
BDC	91 16
BCD	47 54

(4) In the triangle CBD, we have all the angles, and CB and BD, to find CD.

As sin BDC : BC :: sin CBD : CD

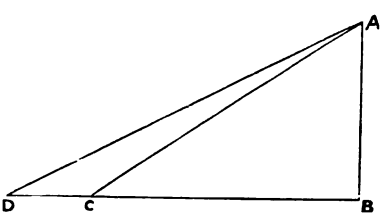
or CD = BC . sin CBD . cosec BDC

$\log BC =$	2·738860
$\log \sin CBD =$	9·815485
$\log \operatorname{cosec} BDC =$	10·000106
CD 358·4 yards =	2·544451

Answer, 358·4 yards.

Question II.—The angle of elevation of a hill, near its base, is 31° 18′, and 214 yards farther off, it is 26° 18′; required, the perpendicular height of the hill, and the distance of the perpendicular from the first station.

Fig. 57.



Let AB represent the perpendicular height of the hill, C the first station, and D the second. Then we have

CD = 214 yards

angle ACB = 31° 18′

„ ADB = 26 18

to find AB and BC.

The angle ACD = (180 − ACB) = (180° − 31° 18′)

„ = 148° 42′

(1) In the triangle ACD, the angles ACD and ADC are known, and the side DC, from which AC may be found.

DAC = 180° − (ACD + ADC)
„ = 180° − (148° 42′ + 26° 18′)
„ = 5°

Then, As sin DAC : DC :: sin ADC : AC
whence, AC = DC . sin ADC . cosec DAC

log DC =	2·330414
log sin ADC =	9·646474
log cosec DAC =	11·059704
AC 1088 =	3·036592

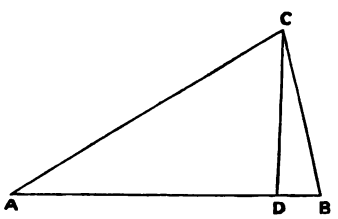
(2) In the right angled triangle ABC, AC and the angle ACB are known, whence AB and BC may be found.

AB = AC . sin ACB		BC = AC . cos ACB	
log AC =	3·036629	log AC =	3·036629
log sin ACB =	9·715602	log cos ACB =	9·931691
AB 565·2 =	2·752231	BC 929·6 =	2·968320

Answer, { Height of hill = 565·2 yards.
Dist of perpendicular = 929·6 yards.

Question III.—Having measured a base of 260 yards in a straight line, close by the side of a river, I found that the two angles, one at each end of the line, subtended by the other end, and a tree close to the opposite bank, were 40° and 80°; what was the breadth of the river?

Fig. 58



We have AB = 260 yards
angle CAB = 40°
„ CBA = 80°
to find CD, the perpendicular from C on AB.

(1) To find BC.

ACB = (180° − CAB + CBA)
= (180° − 40° + 80°)
= 60°

Then, As, sin ACB : AB :: sin CAB : CB
whence, CB = AB . sin CAB . cosec ACB

log AB =	2·414973
log sin CAB =	9·808067
log cosec ACB =	10·062469
CB 192·9 =	2·285509

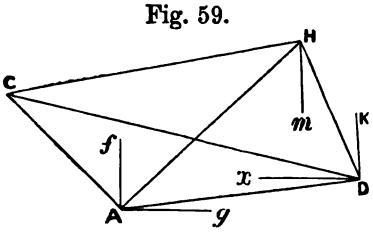
(2) To find DC.

$$\begin{aligned} DC &= BC \cdot \sin CBD \\ \log BC &= 2{\cdot}285509 \\ \log \sin CBD &= 9{\cdot}993351 \\ DC\ 190 &= 2{\cdot}278860 \end{aligned}$$

Answer, 190 yards.

Question IV.—A cape was observed to bear from us NW, and another headland to bear NNE½E. Standing away ENE½E 23 miles we found the first bore from us WNW, and the second N*b*W½W: required the bearing and distance of the cape from the headland.

Fig. 59.



Let A represent the first position of the observer, D the second, C the cape, H the headland, and let H*m*, A*f*, KD, D*x*, and A*g* represent north, south, east and west lines.

Then, we have,

Bearing of cape from A = NW = 4 points = *f*AC

„ headland from A = NNE½E = 2½ points = *f*AH

hence CAH = 6½ points.

And the direction of AD being ENE½E, the angle DA*g* = 1½ points, A*g* being an east line.

Then since *f*A*g* = 8 points, and *f*AH = 2½ and DA*g* = 1½, therefore HAD = 4 points.

and CAD = CA*f*, (4) + *f*AH (2½) + HAD (4) = 10½ points.

Again,

Bearing of cape from D = WNW = 6 points = KDC

„ headland from D = N*b*W½W = 1½ points = KDH

hence CDH = 4½ points.

but KDC = 6 points, hence CD*x* = 2 points

and *x*DA = DA*g* = 1½ points

therefore HDA = 8 points

(1) To find AC

$$\begin{aligned} CAD &= 10\tfrac{1}{2} \text{ points} \\ CDA &= 3\tfrac{1}{2} \text{ „} \\ & \quad 14 \\ & \quad 16 \\ ACD &= 2 \end{aligned}$$

As sin ACD : AD :: sin ADC : AC
or sin $2p$: 23 :: sin $3\frac{1}{2}p$: AC
whence AC = 23 . sin $3\frac{1}{2}$ cosec 2
log 23 = 1·361728
sin $3\frac{1}{2}$ = 9·802359
cosec 2 = 10·417160
AC 38·13 = 1·581247

(2) To find AH

In the triangle HDA, we have HDA = 8 points, (90°):
and AD = 23 : and HAD = 4 points, then
HA = AD . sec HAD
log AD = 1·361728
log sec HAD = 10·150515
AH 32·52 = 1·512243

(3) To find angles ACH and AHC,

In the triangle CAH ; CA = 38·13 ; AH = 32·52
and CAH = $6\frac{1}{2}$ points

as
(AC+AH) : (AC−AH) :: $\tan\frac{1}{2}$(AHC+ACH) : $\tan\frac{1}{2}$(AHC−ACH)
or 70·65 : 5·61 :: tan $4\frac{3}{4}$ points : $\tan\frac{1}{2}$ (H−C)

$$\text{or } \tan\tfrac{1}{2}(H - C) = \frac{5\cdot61 \,.\, \tan 4\frac{3}{4} \text{ points}}{70\cdot65}$$

log 5·61 = 0·748963
log tan $4\frac{3}{4}$ points = 10·129801
10·878764
log 70·65 = 1·849112
$\frac{1}{2}$(H − C) 6° 6′ . . . tan 9·029652
$\frac{1}{2}$(H + C) 53 25 . . . ($4\frac{3}{4}$ points)
AHC 59 31
ACH 47 19

(4) To find CH

As sin ACH : AH :: sin CAH . CH
or sin 47° 19′ : 32·52 :: sin $6\frac{1}{2}$ points : CH
therefore, CH = 32·52 . sin $6\frac{1}{2}$ points . cosec 47° 19′
log 32·52 = 1·512284
log sin $6\frac{1}{2}$ points = 9·980885
log cosec 47° 19′ = 10·133647
CH 42·34 = 1·626816

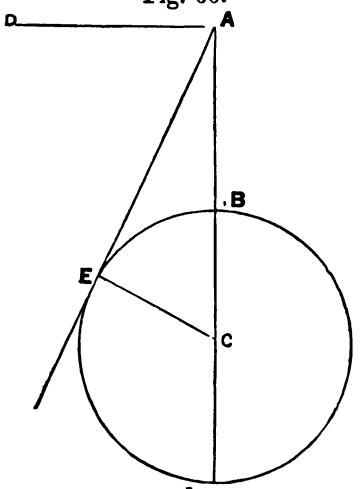
(5) To find the bearing of C from H,

$$\begin{aligned} AHm &= 2\tfrac{1}{2} \text{ points} = 28^\circ\ 7' \\ AHC &= \quad\ 59\ 31 \\ \text{hence } CHm, \text{ the bearing of C from H} &= 87\ 38 \end{aligned}$$

Answer, { Distance CH = 42·34 miles.
Bearing of Cape from Headland S 87°38′ W.

Question V.—From the top of a mountain, 3 miles in height the visible horizon appeared depressed 2° 13′ 27″; required the diameter of the earth, and the distance of the boundary of the visible horizon.

Fig. 60.



In the figure, let BA = h the height of the eye; CE = CB = r; the angle DAE the depression of the horizon = A; and AE the distance of the boundary of the visible horizon = d. Then

$$\frac{AC}{CE} = \frac{r+h}{r} = 1 + \frac{h}{r}$$

but $\frac{AC}{CE} = \sec C$, therefore

$$\sec C = 1 + \frac{h}{r}$$

but C = A (see Investigation for "Dip"), therefore

$$\sec A = 1 + \frac{h}{r};\ \text{or}\ \frac{1}{\cos A} = 1 + \frac{h}{r};\ \text{or}\ \frac{1}{\cos A} - 1 = \frac{h}{r}$$

$$i.e.,\ \frac{1 - \cos A}{\cos A} = \frac{h}{r};\ \text{(call this Equation I)}$$

$$\text{Again, } \frac{EC}{EA} = \cot C;\ i.e.,\ \frac{r}{d} = \frac{\cos C}{\sin C} = \frac{\cos A}{\sin A} \text{(Equation II)}$$

multiplying (Equation I) (by Equation II), we have,

$$\frac{h}{r} \times \frac{r}{d} = \frac{1-\cos A}{\cos A} \times \frac{\cos A}{\sin A} = \frac{1-\cos A}{\sin A} = \frac{2\sin^2 \frac{1}{2}A}{2 \sin \frac{1}{2}A . \cos \frac{1}{2}A} \quad \frac{h}{d} = \tan \tfrac{1}{2}A$$

$$\text{or } h = d \,.\, \tan \tfrac{1}{2} A$$

$$\text{and } d = h \,.\, \cot \tfrac{1}{2} A.$$

$$\log d = \log h + \log \cot \tfrac{1}{2} A$$

$$\log h = \log 3 = 0{\cdot}477121$$
$$\log \cot \tfrac{1}{2} A = \log \cot 1^\circ\ 6'\ 43'' = 11{\cdot}711996$$
$$d\ 154{\cdot}2 = \overline{2{\cdot}189117}$$

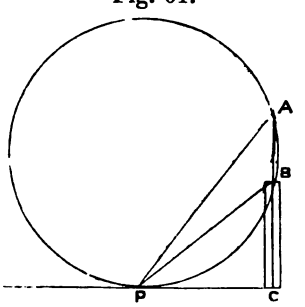
Again, $r = d \cot A$

$$d = 154{\cdot}2 \log = 2{\cdot}188084$$
$$\log \cot A = 11{\cdot}410740$$
$$r = 3970 = \overline{3{\cdot}598824}$$
$$2$$
$$\overline{7940}$$

Answer, { Boundary of horizon 154·2 miles
Diameter of earth (nearly) 7940 miles.

Question VI.—A statue 12 feet high, stands on the top of a column whose height is 48 feet above the level of the eye; at what distance from the base of the column on the same horizontal plane will the statue appear under the greatest possible vertical angle, and what will that angle be?

Fig. 61.



Let BC represent the height of the column 48 feet; AB that of the statue 12 feet; then describe a circle whose circumference shall pass through the two points A and B, touching PC in P; join AP and BP; then APB will be the greatest possible angle under which the statue will appear, on the plane PC.

Then by (Euc. III. 36), since PC touches the circle, and CA cuts it, both lines being drawn from the point C, therefore

$$AC \cdot CB = CP^2$$
$$\text{or } \sqrt{AC \cdot CB} = CP$$
$$i.e., \sqrt{(AB + BC) \cdot BC} = CP$$
$$i.e., \sqrt{(12 + 48) \cdot 48} = CP$$
$$\text{whence } CP = 53{\cdot}7$$

In the triangle ACP

$$\frac{AC}{CP} = \tan APC$$
$$\log AC = 11{\cdot}778151$$
$$\log CP = 1{\cdot}729974$$
$$APC\ 48^\circ\ 10'\ \tan \overline{10{\cdot}048177}$$

In the triangle BCP

$$\frac{BC}{CP} = \tan BPC$$
$$\log BC = 11{\cdot}681241$$
$$\log PC = 1{\cdot}729974$$
$$BPC\ 41^\circ\ 47'\ \tan \overline{9{\cdot}951267}$$

$$\text{And APC} - \text{BPC} = \text{APB}$$

$$\begin{aligned} \text{APC} &= 48^\circ\ 10' \\ \text{BPC} &= 41\ \ 47 \\ \text{APB} &= \ \ 6\ \ 23 \end{aligned}$$

Answer, { Dist 53·7 feet / Angle 6° 23′

(2) *Exercises in Spherical Trigonometry Applied to Astronomy.*

Question VII.—In latitude 50° 48′ N, the altitude of the Sun was 46° 20′ (decreasing), and the Sun's declination was 23° 27′ 45″ N, find the apparent time of observation.

(See fig. in Investigation for Hour Angle).

We have ZP (colatitude) = 39° 12′ 0″
ZS (zen distance) = 43 40 0
PS (polar distance) = 66 32 15

With these elements, it is required to find the angle P, the westerly hour angle of the Sun, which measures the apparent time. (See Definitions Nautical Astronomy).

$$\text{Cos } \tfrac{1}{2}\text{ZPS} = \sqrt{\sin \text{S} \cdot \sin (\text{S} - \text{ZS}) \operatorname{cosec} \text{ZP} \operatorname{cosec} \text{PS}}$$

	° ′ ″		
PS	66. 32. 15	cosec	10·037478
PZ	39. 12. 0	cosec	10·199263
PS	43. 40. 0		
	2)149. 24. 15		
S	74. 42. 7	sin	9·984334
ZS	43. 40. 0		
(S − ZS)	31. 2. 7	sin	9·712280
	° ′ ″	cos	2)19·933355
½ ZPS	22. 9. 33	cos	9·966677
	2		
P	44. 19. 6		
	4		
	6\0)17\7. 16. 24		
(In time)	2. 57. 16·4 PM		

H. M. S.
Answer, 2 57 16·4 PM.

Question VIII.—The altitude of a star, when due east, was 10°, and when due south, 40°; find the latitude.

When an object is on the prime vertical, (*i. e.*, when due east or west), we have

(1) Sin dec = sin alt . sin lat . (See Investigation).

And when on the meridian,

(2) Lat = zd + dec

or lat − zd = dec

and sin (lat − zd) = sin dec

i. e., sin lat . cos zd − cos lat . sin zd = sin dec

Let the altitude when due south = a'

then sin lat . sin a' − cos lat . cos a' = sin dec (3)

But the declinations when due east, and when due south, being considered equal, we have by (1)

Sin dec = sin alt sin lat

and by (3)

Sin dec = sin lat . sin a' − cos lat cos a'

Equating these values,

sin lat . sin a' − cos lat . cos a' = sin a . sin lat

Dividing by sin lat

sin a' − cot lat cos a' = sin a

or sin a' − sin a = cot lat . cos a'

whence, $\frac{\sin a' - \sin a}{\cos a'}$ = cot lat

But, in the question, a = 40°, and a = 10°,

therefore, 2 cos ½ (a' + a) . sin ½ (a' − a) . sec a' = cot lat

or 2 cos 25° . sin 15° . sec 40° = cot lat

log 2 =	0·301030
log cos 25° =	9·957276
log sin 15° =	9·412996
log sec 40° =	10·115746
lat 58° 31′ . cot	9·787048

Question IX.—Given the Sun's altitude at six o'clock, and the hour angle when setting: to find the latitude and declination.

When an object is on the six o'clock hour circle. (See Investigation).

Sin a = sin d . sin l (1)

When setting,

Cos h = tan d . tan l . (2)

or cos h = $\frac{\sin d \,.\, \sin l}{\cos d \,.\, \cos l}$ (3)

Dividing (1) by (3). (The declinations being equal).

$$\frac{\sin a}{\cos h} = \frac{\sin d \,.\, \sin l}{1} \times \frac{\cos d \,.\, \cos l}{\sin d \,.\, \sin l}$$

$$\text{or } \frac{\sin a}{\cos h} = \cos d \,.\, \cos l \ldots \quad (4)$$

$$\text{But } \sin a = \sin d \,.\, \sin l \ldots \quad (1)$$

Adding (1) and (4), we have,

$$\frac{\sin a}{\cos h} + \sin a = \cos d \cos l + \sin d \,.\, \sin l$$

$$\text{and } \frac{\sin a}{\cos h} - \sin a = \cos d \cos l - \sin d \,.\, \sin l, \text{ by}$$

Subtracting (1) from (4)

$$\text{therefore, } \sin a \left\{ \frac{1}{\cos h} + 1 \right\} = \cos (d - l)$$

$$\text{and } \sin a \left\{ \frac{1}{\cos h} - 1 \right\} = \cos (d + l)$$

$$i.e. \quad \sin a \left\{ \frac{1 + \cos h}{\cos h} \right\} = \cos (d - l)$$

$$\text{and } \sin a \left\{ \frac{1 - \cos h}{\cos h} \right\} = \cos (d + l)$$

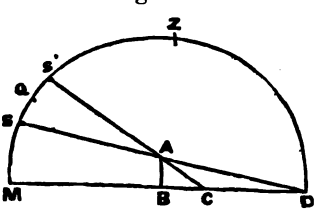
$$\text{or } \sin a \,.\, 2 \cos^2 \tfrac{1}{2}h \,.\, \sec h = \cos (d - l)$$

$$\text{and } \sin a \,.\, 2 \sin^2 \tfrac{1}{2}h \,.\, \sec h = \cos (d + l)$$

and from these two equations, d and l, (the declination and latitude), may be found.

Question X.—At noon on the shortest day, the shadow of a perpendicular stick was 7 times as long, as its shadow at noon, on the longest day: required the latitude, the declination being 23° 28′.

Fig. 62.



Let the circle MZD represent the observer's meridian, Z the zenith, Q the point where the equator meets the meridian. S′ the Sun's place at noon on the longest day (dec 23° 28′ N), and S his place at noon on the shortest day (dec 23° 28′ S) represented by QS′ and QS. Let AB be a perpendicular stick, S′C and SD rays of light, casting the shadows BC, and BD, on each day.

Then, since S'Q = SQ = 23° 28'

Therefore SS' = 46° 56'

Let BC = x, then by Question BD = $7x$

and let BA = y.

Then SS', (46° 56') measures SAS' = CAD

Let ADC = θ, then, since

ACB = CAD + CDA (Euclid, I. 32)

therefore ACB = (46° 56' + θ)

Now, in the right angled triangle ABD,

$$\frac{y}{7x} = \tan\theta$$

and in the right angled triangle ABC

$$\frac{y}{x} = \tan \mathrm{ACB} = \tan(46° 56' + \theta)$$

therefore,

$$\frac{\tan\theta}{\tan(46° 56' + \theta)} = \frac{1}{7}$$

$$\text{or } 7\tan\theta = \tan(46° 56' + \theta) = \frac{\tan 46° 56' + \tan\theta}{1 - \tan 46° 56' \tan\theta}$$

whence we have,

$$7\tan\theta - 7\tan 46° 56' \tan^2\theta = \tan 46° 56' + \tan\theta$$

$$\text{or, } \tan^2\theta - \frac{6}{7\tan 46° 56'}.\tan\theta = -\frac{1}{7}$$

and

$$\tan^2\theta - \frac{6}{7\tan 46° 56'}\tan\theta + \frac{9}{49\tan^2 46° 56'} = -\frac{1}{7} + \frac{9}{49\tan^2 46° 56'}$$

$$= \frac{9 - 7\tan^2 46° 56'}{49\tan^2 46° 56'}$$

whence,

$$\tan\theta = \frac{\sqrt{9 - 7\tan^2 46° 56'} - 3}{7\tan 46° 56'}$$

The value of $7\tan^2 47° 56'$ may be found by logs to be = 8·011; and that of $7\tan 46° 56'$ to be 7·489

$$\text{hence } \tan\theta \frac{\sqrt{9 - 8·011} + 3}{7·489} = \frac{4}{7·489} \text{ nearly}$$

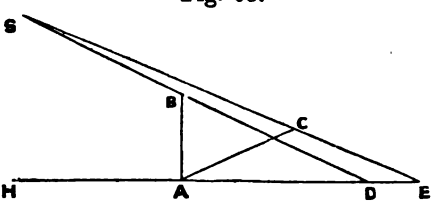
from which θ = 28° 6' 26"

But (θ, the angle CDA) is measured by the opposite arc SM, which is the Sun's altitude on the shortest day, at noon,

	°	′	″
Then since mer alt =	28.	6.	26
mer zen dist =	61.	53.	34
declination =	23.	28.	0
whence latitude =	38.	25.	34 nearly.

Question XI.—The length of the shadow of a perpendicular object was 4 feet, and its longest, when sloping, was 5 feet; required the Sun's altitude.

Fig. 63.



Let AB represent the object when perpendicular, and AC when sloping; SBD and SCE rays of light from the Sun at S. These lines may be considered parallel, as coming from a great distance, and cutting AC at right angles, and making the angles SDH and SEH equal.

AD (1st shadow) = 4 feet

AE (2nd „) = 5 „

and SDH = SEH = Sun's altitude.

Then $\frac{BA}{AD}$ = tan BDA, or $\frac{BA}{4}$ = tan alt

and $\frac{CA}{AE}$ = sin CEA, or $\frac{CA}{5}$ = sin alt

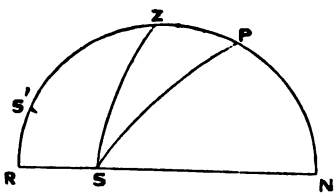
Therefore, $\frac{\sin \text{ alt}}{\tan \text{ alt}} = \frac{CA}{5} \times \frac{4}{BA} = \frac{4}{5}$

or $\cos \text{ alt} = \frac{4}{5}$, whence

alt = 36° 52′ 15″.

Question XII.—In latitude 45° N, the meridian altitude of the Sun was 30°: show that the tangent of quarter the length of the day, $= \frac{1}{\sqrt[4]{3}}$.

Fig. 64.



Let RN represent the horizon; RZPN the meridian; Z the zenith; P the pole; S the Sun's rising point; PS his polar distance; and ZS his zenith distance (= 90°). Then the angle SPS', the interval between his rising and coming to the meridian = ½ the length of the day.

$$\text{Cos ZPS } (h) = \frac{\cos \text{ZS} - \cos \text{ZP} \,.\, \cos \text{PS}}{\sin \text{ZP} \sin \text{PS}}$$

and since ZS = 90°

$$\cos h = \cot \text{ZP} \cot \text{PS}$$

$$= \tan \text{lat} \,.\, \cot p$$

Now since lat = zd − dec

$$45° = 60° - \text{dec}$$

whence dec = 15° and p = 105°

therefore, $\cos h = -\tan 45° \cot 105° = -1 \,.\, \times - \cot 75°$

$$= + \cot 75°$$

or $\sec h = \tan 75° = \tan (45 + 30)$

$$= \frac{1 + \tan 30}{1 - \tan 30} = \frac{1 + \frac{1}{3}\sqrt{3}}{1 - \frac{1}{3}\sqrt{3}}$$

$$\text{or } \frac{\sec h - 1}{\sec h + 1} = \frac{\frac{1}{3}\sqrt{3}}{2} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

i.e.

$$\frac{\frac{1}{\cos h} - 1}{\frac{1}{\cos h} + 1} = \frac{1}{\sqrt{3}}; \text{ or } \frac{1 - \cos h}{1 + \cos h} = \frac{1}{\sqrt{3}}$$

i.e.

$$\frac{2 \sin \frac{2h}{2}}{2 \cos \frac{2h}{2}} = \frac{1}{\sqrt{3}} \text{ whence } \tan \frac{h}{2} = \frac{1}{\sqrt[4]{3}}$$

or $\tan \frac{1}{4}$ length of the day $= \frac{1}{\sqrt[4]{3}}$.

NOTE. These questions are to be found in "Jean's Trigonometry" (Part I). They are inserted, being likely to prove useful in the Solutions of the Miscellaneous Examples at the end of this Work.

EXAMPLES FOR EXERCISE.

Right Angled Plane Triangles.

In the right angled plane triangle ABC, given

1. AB = 365·2 : A = 47° 30′ 15″ : find the rest.
   Ans. AC 540·6 : BC 398·6 : C 42° 29′ 45″
2. BC = 395·4 : A = 39° 15′ 40″
   Ans. AC 624·9 : AB 485·8 : 50° 44′ 20″
3. AC = 360·7 : C = 51° 26′ 15″
   Ans. AB 282 : BC 224·8 : A 38° 33′ 45″
4. AB = 31·15 : C 29° 10′ 15″
   Ans. AC 63·9 : BC 55·8 : A 60° 49′ 45″
5. BC = 16·95 : C 69° 10′ 30″
   Ans. AC 47·67 : AB 44·55 : A 20° 49′ 30″
6. AC = 12·25 : BC = 6·95
   Ans. AB 10·08 : A 34° 33′ 55″
7. AC = 26·05 : AB = 2·97
   Ans. BC 25·88 : A 83° 27′ 12″
8. AB = 81·256 : BC = 30·1
   Ans. AC 86·65 : A 20° 19′ 34″
9. AC = $21\frac{19}{40}$: A = 31° 16′ 20″
   Ans. BC 11·16 : AB 18·35
10. AB = $31\frac{11}{14}$: C = 69° 31′ 10″
    Ans. BC 11·87 : AC 33·95
11. AC = $11\frac{31}{45}$: C = 29° 10′ 40″
    Ans. AB 5·7 : BC 10·19
12. AB = $11\frac{11}{51}$: A = 31° 45′ 15″
    Ans. AC 13·18 : BC 6·96
13. AC = $112\frac{16}{25}$: A = 47° 20′ 10″
    Ans. AB 76·24 : BC 82·72
14. AC = $602\frac{2}{5}$: AB = $51\frac{14}{25}$
    Ans. BC 600 : A 85° 5′ 18″
15. BC = $21\frac{17}{40}$: AB = $69\frac{19}{21}$
    Ans. AC 72·72 : A 17° 8′ 10″

Oblique Angled Triangles.

CASE I.

In the oblique angled plane triangle ABC, given

1. AC = 23·01 : AB = 19·42 : B = 39° 42′ 15″ : find the other parts

Ans. ∠C 32° 37′ 24″ : A 107° 40′ 24″ : BC 34·32

2. AB = 12·29 : BC = 14·21 : A = 31° 16′ 17″

Ans. C 26° 49′ 35″ : B 122° 3′ 8″ : AC 23·2

3. AC = 31·25 : BC = 21·05 : B = 26° 4′ 17″

Ans. C 136° 26′ 12″ : A 17° 19′ 31″ : AB 48·71

4. AB = 11·275 : AC = 21·342 : B = 101° 31′ 29

Ans. C 31° 10′ 28″ : A 47° 18′ 3″ : BC 16·01

5. AB = 112·254 : BC = 16·175 : C = 121° 16′ 15″

Ans. A 7° 4′ 27″ : B 51° 39′ 18″ : AC 103

6. AC = 31·256 : BC = 69·175 : A = 139° 45′ 20″

Ans. B 16° 58′ 20″ : C 23° 16′ 20″ : AB 42·3

7. A = 21° 16′ 15″ : B = 39° 44′ 27″ : AB = 69·25

Ans. BC 28·72 : AC 50·62 : C 118° 59′ 18″

8. B = 39° 44′ 27″ : C = 68° 21′ 15″ : BC = 31·14

Ans. AC 21·06 : AB 30·83 : A 70° 54′ 8″

9. C = 21° 14′ 15″ : B = 39° 22′ 10″ : BC = 61·152

Ans. AC 44·51 : AB 25·42 : A 119° 23′ 35″

10. A = 31° 26′ 15″ : B = 21° 17′ 15″ : AC = 69·25

Ans. AB 151·8 : BC 99·5 : C 127° 16′ 30″

11. B = 61° 27′ 12″ : A = 39° 44′ 26″ : AC = 39·152

Ans. AB 43·75 : BC 28·5 : C 78° 58′ 22″

12. C = 69° 44′ 25″ : B = 69° 44′ 25′ : AC = 30·52

Ans. AB 30·52 : BC 21·13 : A 40° 31′ 10″

CASE II.

In the oblique angled plane triangle ABC, given

1. AB = 21·65 : BC = 29·85 : B = 101° 31′ 25″ : find the other parts

Ans. A 47° 10′ 13″ : C 31° 18′ 21″ : AC 39·88

2. AC = 161·5 : AB = 214·8 : A = 39° 16′ 25″

Ans. B 48° 42′ 42″ : C 92° 0′ 52″ : BC 136

3. AB = 61·25 : BC = 41·85 : B = 121° 15′ 0″

Ans. A 23° 19′ 44″ : C 35° 25′ 16″ AC 90·35

4. AC = 612·25 : BC = 31·145 : C = 31° 10′ 40″
   Ans. A 1° 34′ 38″ : B 147° 14′ 42″ : AB 585·7
5. AB = 12·85 : AC = 12·85 : A = 126° 16′ 15″
   Ans. B 26° 51′ 52″ : C 26° 51′ 52″ : BC 22·92
6. AC = 216·85 : AB = 314·75 : A = 112° 13′ 45″
   Ans. B 26° 50′ 5″ : C 40° 56′ 9″ : BC 444·7
7. BC = 16·152 : AC = 17·152 : C = 29° 45′ 30″
   Ans. A 68° 40′ 20″ : B 81° 34′ 10″ : AB 8·606
8. AB = 12·625 : BC = 16·725 : B = 116° 31′ 10″
   Ans. A 36° 40′ 44″ : C 26° 48′ 6″ : AC 25·05
9. BC = 21·65 : AC = 19·25 : C = 112° 42′ 30″
   Ans. A 35° 52′ 56″ : B 31° 24′ 34″ : AB 34·08
10. AB = 16·95 : BC = 25·92 : B = 116° 44′ 35″
    Ans. A 38° 58′ 17″ : C 24° 17′ 7″ : AC 36·8

CASE III.

In the oblique angled plane triangle ABC, given

1. AC = 20·75 : AB = 30·62 : BC = 40·12 : find A
   Ans. A 100° 0′ 6″
2. AC = 12·25 : AB = 16·45 : BC = 10·12 : find B
   Ans. B 47° 57′ 40″
3. AC = 8·645 : AB = 6·125 : BC = 12·95 : find C
   Ans. C 23° 45′ 50″
4. AC = 24·87 : AB = 25·67 : BC = 31·25 : find A
   Ans. A 76° 24′ 56″
5. AC = 39·26 : AB = 31·27 : BC = 29·25 : find B
   Ans. B 80° 48′ 52″
6. AC = 67·15 : AB = 39·25 : BC = 64·15 : find C
   Ans. C 30° 11′ 24″
7. AC = 1·615 : AB = 2·245 : BC = 3·125 : find A
   Ans. A 107° 2′ 32″
8. AC = 10·16 : AB = 6·14 : BC = 12·18 : find B
   Ans. B 56° 22′ 28″
9. AC = 39·62 : AB = 21·64 : BC = 27·28 : find C
   Ans. C 31° 22′ 14″
10. AC = $2\frac{3}{4}$: AB = $4\frac{1}{47}$: BC = $6\frac{13}{19}$: find A
    Ans. A 119° 52′ 40″

11. AC = ·164 : AB = ·265 : BC = ·314 : find B
Ans. B 168° 58′ 26″
12. AC = ·029 : AB = ·015 : BC = ·036 : find C
Ans. C 23° 41′ 36″

To Find the Area.

In any plane triangle ABC, given
1. Base = 30·25 : perpendicular = 10·5
Area 158·8125
2. Base = 60 : perpendicular = 10·5
Area 315
3. AC = 11·25 : AB = 10·75 : A = 22° 30′
Area 23·15
4. AB = 20·16 : BC = 17·15 : B = 62° 10′
Area 102·85
5. AC = 11·72 : BC = 10·45 : C = 61° 10′
Area 53·6
6. AB = 10·45 : AC = 6·25 : A = 39° 17′
Area 20·67
7. AC = 6·25 : AB = 10·95 : A 11° 25′
Area 6·68
8. AB = 3·245 : BC = 6·125 : B = 29° 4′
Area 4·83
9. BC = 12·62 : AB = 7·25 : B = 110° 40′
Area 42·8
10. AB = 12·42 : BC = 15·24 : AC = 11·27
Area 68·81
11. AB = 11·64 : BC = 21·75 : AC = 18·45
Area 107·3
12. AB = 9·65 : BC = 10·72 : AC = 12·42
Area 49·86

Spherical Triangles.

case i.

In an oblique angled spherical triangle ABC, given,
1. AC = 120° 15′ 30″ : CB = 110° 45′ 15″ : C = 45° 20′ 30″ : find AB
Ans. 41° 43′ 50″

2. BC = 64° 12′ 35″ : AC = 21° 14′ 15″ : C = 12° 45′ 30″ : find AB

Ans. 43° 38′ 42″

3. AB = 71° 15′ 15″ : AC = 39° 44′ 30″ : A = 112° 44′ 15″ : find BC

Ans. 89° 14′ 52″

4. AB = 26° 45′ 10″ : BC = 39° 51′ 44″ : B = 61° 20′ 25″ : find AC

Ans. 34° 31′ 58″

5. BC = 39° 11′ 44″ : AC = 69° 21′ 54″ : C = 11° 44′ 15″ : find AB

Ans. 31° 33′ 4″

6. AC = 110° 44′ 30″ : AB = 116° 41′ 10″ : A = 110° 15′ 40″ : find BC

Ans. 97° 29′ 14″

7. AB = 12° 44′ 30″ : BC = 16° 31′ 10″ : B = 29° 44′ 30″ : find AC

Ans. 8° 17′ 4″

8. AB = 69° 29′ 54″ : BC = 71° 10′ 44″ : B = 61° 12′ 45″ : find AC

Ans. 57° 19′ 16″

9. AC = 112° 44′ 30″ : CB = 116° 17′ 40″ : C = 110° 15′ 0″ : find AB

Ans. 96° 36′ 4″

10. AB = 91° 30′ 15″ : BC = 102° 15′ 16″ : B = 39° 15′ 10″ : find AC

Ans. 40° 21′ 26″

11. AC = 101° 15′ 30″ : AB = 102° 14′ 20″ : A = 12° 39′ 15″ : find BC

Ans. 12° 25′ 28″

12. BC = 69° 21′ 10″ : AC = 72° 30′ 50″ : C = 112° 15′ 45″ : find AB

Ans. 103° 25′ 30″

CASE II.

In the oblique angled spherical triangle ABC, given

1. AB = 21° 14′ 30″ : BC = 39° 20′ 10″ : B = 60° 31′ 15″ : to find angles A and C

Ans. A 91° 5′ 31″ : C 34° 51′ 11″

2. BC = 12° 40′ 30″ : AC = 15° 50′ 35″ : C = 29° 30′ 10″ : to find angles A and B

Ans. A 52° 35′ 43″ : B 98° 45′ 45″

3. AB = 39° 10′ 20″ : AC = 16° 14′ 34″ : A = 31° 12′ 40″ : o find angles B and C

Ans. B 18° 59′ 54″ : C 132° 40′ 46″

4. AC = 21° 15′ 40″ : CB = 34° 12′ 40″ : C = 71° 15′ 10″ : to find angles A and B

Ans. A 76° 7′ 46″ : B 38° 45′ 58″

5. AB = 34° 10′ 20″ : AC = 24° 15′ 10″ : A = 31° 15′ 40″ : to find angles B and C

Ans. B 43° 53′ 15″ : C 108° 34′ 3″

6. BC = 21° 15′ 10″ : AC = 39° 12′ 25″ : C = 44° 16′ 30″ : to find angles A and B

Ans. A 33° 6′ 38″ : B 107° 42′ 44″

CASE III.

In the oblique angled spherical triangle ABC, given

1 AB = 12° 40′ 30″ : BC = 21° 14′ 15″ : AC = 16° 17′ 42″ : to find A

Ans. 94° 3′ 20″

2. AB = 39° 15′ 40″ : BC = 45° 10′ 20″ : AC = 50° 21′ 15″ : to find B

Ans. 78° 8′ 50″

3. AB = 61° 17′ 20″ : BC = 72° 14′ 15″ : AC = 70° 20′ 10″ : to find C

Ans. 65° 5′ 20″

4. AB = 42° 40′ 40″ : BC = 51° 27′ 20″ : = AC 40° 31′ 30″ : to find A

Ans. 81° 36′ 20″

5. AB = 69° 51′ 20″ : BC = 110° 14′ 20″ : AC = 81° 16′ 40″ : to find B

Ans. 72° 5′ 46″

6. AB = 110° 39′ 50″ : BC = 114° 50′ 44″ : AC = 111° 32′ 10″ : to find C

Ans. 126° 55′ 35″

CASE IV.

In the oblique angled spherical triangle ABC, given

1. A = 31° 20′ 10″ : B = 34° 27′ 20″ : AB = 62° 40′ 40″ : find AC and BC

Ans. AC 37° 41′ 18″ : BC 34° 11′ 32″

2. B = 72° 10′ 53″ : C = 69° 31′ 10″ : BC = 51° 20′ 10″ : find AB and AC

Ans. AB 54° 59′ 56″ : AC 56° 21′ 10″

3. C = 64° 12′ 20″ : A = 61° 20′ 15″ : AC = 72° 15′ 50″ : find AB and BC

Ans. AB 59° 5′ 55″ : BC 56° 44′ 21″

4. A = 79° 40′ 15″ : B = 72° 15′ 15″ : AB = 60° 40′ 20″ : find AC and BC

Ans. AC 65° 12′ 36″ : BC 69° 40′ 42″

5. B = 62° 40′ 25″ : C = 75° 10′ 35″ : BC = 72° 12′ 40″ : find AC and AB

Ans. AC 58° 45′ 18″ : AB 68° 29′ 5″

6. C = 29° 47′ 20″ : A = 35° 10′ 40″ : AC = 72° 40′ 10″ : find AB and BC

Ans. AB 37° 22′ 15″ : BC 44° 44′ 17″

CASE V.

In the oblique angled spherical triangle ABC, given

1. AC = 39° 40′ 15″ : BC = 62° 40′ 10″ : = A 59° 20′ 10″ : to find B

Ans. 38° 10′ 42″

2. BC = 72° 40′ 20″ : AB = 69° 31′ 50″ : A = 49° 10′ 20″ : to find C

Ans. 47° 57′ 13″

3. AB = 67° 12′ 50″ : AC = 77° 15′ 40″ : B 39° 40′ 10″ : to find C

Ans. 37° 6′ 40″

4. AC = 62° 12′ 50″ : BC 51° 10′ 40″ : B 41° 10′ 40″ : to find A

Ans. 35° 26′ 14″

CASE VI.

In the oblique angled spherical triangle ABC, given

1. A = 110° 42′ 50″ : B 130° 16′ 40″ : C 95° 10′ 20″ : to find BC

Ans. 112° 52′ 50″

2. A = 75° 24′ 40″ : B 72° 9′ 30″ : C 114° 12′ 30″ : to find AC

Ans. 76° 41′ 48″

3. A = 112° 15′ 20″ : B 116° 17′ 40″ : C 110° 24′ 50″ : to find AB

Ans. 102° 36′ 6″

4. A = 72° 10′ 50″ : B 119° 50′ 20″ : C 110° 40′ 20″ : to find BC

Ans. 53° 35′ 40″

5. A = 112° 42′ 25″ : B 119° 52′ 50″ : C 120° 0′ 0″ : to find AC

Ans. 112° 27′ 28″

6. A = 145° 16′ 10″ : B 120° 17′ 40″ : C 119° 44′ 50″ : to find AB

Ans. 99° 32′ 58″

CASE VII.

Right Angled Spherical Triangles.

In the right angled spherical triangle ABC, (B = 90°), given

1. AC = 118° 24′ 10″ : A 72° 10′ 40″ : find the other parts
   Ans. AB 150° 29′ 30″ : BC 56° 52′ 5″ : C 145° 56′ 47″
2. AC = 110° 20′ 0″ : AB = 59° 50′ 0″
   Ans. A 129° 36′ 37″: C 67° 13′ 25″ : BC 133° 44′ 51″
3. AC = 62° 40′ 10″ : BC = 51° 19′ 30″
   Ans. A 61° 29′ 52″ : C 49° 47′ 4″ : AB 42° 43′ 1″
4. AB = 110° 40′ 30″ : BC = 92° 5′ 0″
   Ans. A 91° 56′ 58″: C 110° 39′ 45″ : AC 89° 15′ 52″
5. AB = 42° 40′ 0″ : C = 51° 50′ 0″
   Ans. A 57° 10′ 56″ : BC 46° 25′ 19″ : AC 59° 32′ 38″
6. BC = 72° 10′ : A = 81° 40′
   Ans. C 28° 14′ 44″ : AB 27° 5′ 7″ : AC 74° 10′ 38″
7. AB = 167° 10′ 30″ : A = 42° 15′ 50″
   Ans. C 130° 58′ 41″: AC 162° 54′ 7″ : BC 11° 24′ 18″
8. AB = 110° 30′ 0″ : BC = 67° 10′ 0″
   Ans. A 68° 28′ 36″: C 109° 0′ 48″ : AC 97° 48′ 38″

PROBLEMS IN TRIGONOMETRY.

1. At the distance of 880 feet from the base of a tower, the angle of elevation of its top, was, 23° 15′ 40″, the height of the observer's eye being 6 feet: required the height of the tower.

Ans. 384·2 feet.

2. From the top of a ship's mast, the angle of depression of another ship's hull 580 yards off was 4° 55′ 30″: find the height of the mast. Ans. 49·97 yards.

3. The angle of elevation of a tower, was 23° 14′ 30″: and 500 yards farther off in the same direction from its base, its elevation was 10° 59′ 50″: required its height.

Ans. 177·5 yards.

4. The angle between two trees on the opposite bank of a river was 62° 30′ 10″, and my distance from either tree was 840, and 920 yards: required the breadth of the river.

Ans. 749 yards.

5. At a certain distance from the base of a tower, the angle of elevation of its top, was double its angle of elevation at three times that distance from its base: find the first angle, and the height of the tower.

Ans. height = dist $\times\sqrt{3}$: angle 60°.

6. The intervals between the flashes and reports of two cannon on a straight fortification, were 4 and 6 sec^{s}, and the angle between the positions of the cannon was 20° 45′: find my shortest distance to the line of fortification (sound travelling 1142 feet in a second).

Ans. 3641 feet.

7. From each of two opposite sides A and B of a street, I took the angles of elevation of a steeple 200 feet high, which could be seen beyond the end of the street, and found them to be, at A, 45° 10′, and at B, 42° 40′. I also measured at A, the horizontal angle, between B and the steeple, 62° 30′: find the breadth of the street. Ans. 218 feet.

8. From a ship at A, the bearing of an island C, was due N, and of another, D, ENE. After sailing due E 10 miles to B, the bearing of the island C was NW$\frac{1}{2}$N, and of D, NNE$\frac{1}{2}$E: find the distance of the islands from each other.

Ans. 16·06 miles.

9. The bearing of a tower from A was NE, and the angle of elevation of its top, was 30°. After walking SE 500 yards to B, it bore from me N$\frac{1}{2}$W: find its height.

Ans. 192·8 yards.

10. The angle of elevation of a tower, bearing N, at a distance a from its base, was 2θ, and after walking due E to a distance c, the angle of elevation was θ; find an expression for θ.

Ans. $\tan\theta = \sqrt{\dfrac{\sqrt{a^2+c^2}+2a}{\sqrt{a^2+c^2}}}$

11. At a distance a from the base of a tower, its angle of elevation was 2θ. I then measured b feet farther off in same direction as the distance a, and walked round the circumference of a circle, of which the base of the tower was the centre, till the elevation was θ; show that the height of the tower may be represented by $\sqrt{b^2-a^2}$.

12. Standing in a line between two elevations, at a feet from the lower, and b from the higher, I found that they subtended equal angles, and that the squares of their heights were proportional to the squares of the sum and difference of their distances, show that the heights are to each other as, $a^2(a+b) : b^2(a-b)$.

13. The angle of elevation of the summit of a hill, at a feet from its base was 60°, and b feet farther off in a straight line, the angle of elevation of the top of a tower upon it was 30°; find the height of the tower.

Ans. $\dfrac{b-2a}{\sqrt{3}}$

14. The angle of elevation of the top of an eminence, bearing due E., at a feet from its base was 2θ, and b feet farther south, the angle was θ; find its height.

Ans. $\sqrt{(a^2+b^2)-2a\sqrt{a^2+b^2}}$

15. On a certain day, at noon, when the Sun's declination was 22° 30′ S, the shadow of a perpendicular stick was 7 times the length of its shadow on another day, when the declination was 22° 30′ N.: find the altitude of the Sun at noon on each day, and the latitude of the place.

Ans. { Altitudes 57° 47′ 21″ and 12 47 21 Latitude 54 42 39

16. At noon, on a certain day, when the Sun's declination was 12° 30′ N., the length of the shadow of a perpendicular stick was 5 feet, and its longest shadow, when sloping, 7 feet; required the latitude. Ans. 58° 5′ 5″ N.

17. In latitude 30° N., the meridian altitude of the Sun was 75°; show that the tangent of quarter the length of the day $= \left(\dfrac{2}{\sqrt{3}+1}\right)^{\frac{1}{2}}$

18. Given the Sun's altitude when on the prime vertical 15° 20′, and at 6 o'clock, 13° 22′; find the latitude and declination.

Ans. Lat 69° 13′ 48″; dec 14° 18′ 53″.

19. If $(\alpha + \beta)$ represent the right ascension of an object; $(\alpha - \beta$ its declination; and θ its latitude; show that the cosine of its longitude $= \dfrac{\cos^2 \alpha - \sin^2 \beta}{\cos \theta}$.

20. A ship sails from lat 34° 30′ S, long 25° 20′ W, to lat 39° 10′ S, on a SSW½W course; find her long in.

Ans. 28° 27′ W.

21. A ship sails from lat 67° 30′ N, till her dep is 110 miles, and her D long 260; find her lat in.

Ans. 62° 26′ N.

22. When the Sun's declination was 18° N, he set at $7^h\ 9^m\ 20^s$ P.M.; find the latitude.

Ans. 42° 31′ 4″ N.

23. When the Sun's declination was 20° N, he set 3 hours later than when it was 20° S; find the latitude, and the times of his setting.

Ans. { Lat 46° 26′ N.
Time $7^h\ 30^m$ P.M.
„ $4^h\ 30^m$ P.M.

24. Given the altitudes of Vega and Altair, 85° 50′, and 57° 3′ respectively; their right ascensions, $18^h\ 32^m$ and $19^h\ 44^m$; and their declinations, 38° 40′ N, and 8° 31′ N; find the latitude.

Ans. 39° 26′ 10″ N.

25. Two posts are in a line with the Sun, and are 12 feet apart; the line joining their tops measures 15 feet. I observed that their shadow cast by the Sun extended 6 feet beyond the smaller post; find their heights, and the Sun's altitude.

Ans. { Heights, 13·5 and 4·5 feet.
Altitude of Sun 36° 52′.

26. To determine the height of a column, I set up a staff 10 feet high, in a line with it and the Sun, and found that the end of the shadow extended beyond the staff, to half the distance between it and the column; find its height.

Ans. 30 feet.

27. The hull of a ship, 15 miles off, is just visible upon the horizon; find the height of my eye above the sea.

Ans. 150 feet.

28. The angle of elevation of the top of a tower, a feet high, was 30°, and the vertical angle subtended by a flag staff on the tower from the same point was 5°: find the height of the staff.

Ans. ·2126 a.

29. A pole is inclined to the horizon at an angle of 60°, and at a feet distance on the opposite side to which it inclines, the angle of elevation of its top is 15°: find the height of its top from the ground.

Ans. $\frac{1}{4}a\,(3 - \sqrt{3})$.

30. The mainmasts of two vessels, 180 and 150 feet high, subtended angles of 1° 30′, and 2° 30′ respectively; and the bearings of the vessels, from the same point, were NNW, and NE: find their distance apart.

Ans. 6401 feet.

31. The bearing of a balloon from a certain point was N60°E, and at a distance of a yards farther south its bearing was N30°E, and its angle of elevation was 30°: find its height.

Ans. a, yards.

32. In a certain latitude, when the Sun's declination was 19° 10′ N, he set 6 hours later than when it was 21° 10′ S: find the times of his setting, and the latitude of the place.

Ans. { Times, $9^h\ 12^m\ 20^s$ P.M.
,, $3^h\ 12^m\ 20^s$ P.M.
Latitude, 62° 30′ N.

33. At 3 P.M., when the Sun's declination was 0°; his altitude was 44° 40′: find the latitude of the place.

Ans. 6° 11′ 30″.

34. Given, the altitude and declination of a celestial object, each 45°, and its azimuth 60°; find the latitude and hour angle.

Ans. { Latitude, 36° 52′ 12″
Hour angle, $4^h\ 0^m\ 0^s$.

35. In latitude 45° N, the sum of the altitudes of two bodies on the prime vertical and six o'clock hour circles was 90°, and their declinations were equal: find their altitudes and declination.

Ans. { Altitude of body on prime vertical, 63° 26′ 5″
,, on six o'clock hour circle, 26° 33′ 55″
Declination, 39° 14′ 0″ N.

36. When a certain object was 4^h from the meridian, in lat 45°, its polar distance was equal to the latitude: find its altitude.

Ans. $\sin a = \frac{3}{4}$: 48° 35′ 25″.

37. When a certain object was $2^h\ 14^m\ 48^s$ from the meridian, its altitude was 60°, and its polar distance was equal to the latitude: find the latitude.

$$\text{Ans. } \sin 2l = \frac{\sin 60^\circ}{\cos^2 \frac{1}{2}h} : \text{lat} : 35^\circ\ 29'\ 44''.$$

38. The azimuth of a celestial object was 60°, its polar distance was 60°, and its altitude was equal to the latitude of the place: find the latitude.

$$\text{Ans. } \cos \text{lat} = \frac{1}{\sqrt{3}} : \text{lat} : 54^\circ\ 44'.$$

39. Find the declination of an object whose amplitude is equal to the latitude of the place of observation.

$$\text{Ans. } \sin \text{dec} = \tfrac{1}{2} \,.\, \sin 2 \text{ lat.}$$

40. Where would an object be whose declination is equal to the colatitude of the place?

Ans. in the north point.

41. Show that the latitude of the place is the complement of half the sum of the meridian altitudes of the Sun, at his greatest N and S declination, or when his declination N and S are equal.

42. If a be the meridian altitude of the Sun on June 21st, and a' that on December 21st, show that his declination $= \frac{1}{2}(a - a')$.

43. If a be the Sun's meridian altitude when his declination is N, and a' that when his S declination is double the N, show that N dec $= \frac{1}{3}(a - a')$.

44. If the Sun's meridian altitude on the longest day be a, and on the shortest, a', show that,

$$\tan \text{lat} = \cot \tfrac{1}{2}(a + a').$$

45. If $(a - b)$, be the hour angle of an object when on the prime vertical, and $(a + b)$, its hour angle when setting, show that,

$$\sec 2\, l = \cot a \cot b.$$

46. If $(a - b)$ be the altitude of an object when on the prime vertical and $(a + b)$ its meridian altitude, show that,

$$\tan \text{lat} = \frac{\cos (a + b)\,.}{2 \cos a \,.\, \sin b\,.}$$

47. The meridian distance of two places in the same latitude, is $(a - b)$, and of two others, in a less latitude $(a + b)$: the sum of the latitudes is 90°: find an expression for the difference of latitude.

$$\text{Ans. } \tan \tfrac{1}{2}(l - l') = \frac{b}{a}.$$

48. In the preceding question, if $a = 60$ and $b = 30$ miles: find the latitudes of the parallels.

$$\text{Ans.} \begin{cases} l\ 71^\circ\ 33'\ 54'' \\ l'\ 18\ \ 26\ \ 6 \end{cases}$$

49. The meridian distance of two places, is double that of two other places in a higher latitude; and the latitudes are, as 1 to 2: find them.

$$\text{Ans. } \cos l = \frac{1 + \sqrt{33}}{8}$$

whence lat: $32^\circ\ 29'\ 18''$
and $64\ \ 58\ \ 36$

50. If one latitude be three times the other, and the meridian distances as 1 to 3.

$$\text{Ans. } \cos l = \sqrt{\tfrac{5}{8}}$$

whence lat: $24^\circ\ \ 5'\ 41''$
„ and: $72\ \ 17\ \ 3$

51. Two ships having made the same diff: long:, sail until their departures are as 1 to 2, and their middle latitudes as 3 to 1: find their middle latitudes.

$$\text{Ans.} \begin{cases} 20^\circ\ 42'\ 18'' \\ 62\ \ 6\ 54 \end{cases}$$

52. Two ships sail S45°W, and S60°W, having made the same diff: long: their diff: lats: are as 1 to 2, and their middle latitudes as 3 to 1: find them.

$$\text{Ans.} \begin{cases} 24^\circ\ 56'\ 10'' \\ 74\ \ 48\ \ 30 \end{cases}$$

53. The great circle tracks of two ships cross each other in the vertex of both. The first ship sails to a higher latitude, from lat: 45°, and the second to a higher latitude from lat 60°; their differences of longitude are as 3 to 1: find their differences of longitude.

$$\text{Ans.} \begin{cases} 19^\circ\ \ 8'\ 23'' \\ 57\ \ 25\ \ 9 \end{cases}$$

54. Given the base of a triangle, $2a$: one of the other sides, a, and the angle opposite, 30°: find the area of the triangle.

$$\text{Ans.} \frac{a^2}{2} \cdot \sqrt{3}.$$

55. Given the base of a triangle, $4a$: the angle opposite, 60°, and one of the other sides, $3a$: find the area.

$$\text{Ans. } 5{\cdot}9\ a^2.$$

56. From the hypothenuse of a right angled triangle, whose perpendicular is 6, and base 10, a line is drawn at right angles to the latter, cutting off a triangle, whose area is equal to half that of the original triangle: find its sides.

Ans. $3\sqrt{2}$, $5\sqrt{2}$ and $2\sqrt{17}$.

57. In two triangles, the area of one of which, is double that of the other, the corresponding sides b and a of the larger, and e and d of the smaller, contain angles of 60° and 30° respectively: if $a = d$, show that $b : e :: 2 : \sqrt{3}$.

58. Two ships sailing from the same meridian, meet each other. One sails Sθ°W, the other N2θ°W: the distances sailed are as 7 to $4\frac{1}{4}$: show that their d: lat s: are to each other, as 5·8 : 1·51

59. Two ships sail from the same place, one Sx°W, 40 miles, the other, S3x°E, 50 miles, and reach the same parallel: find their courses, and difference of latitude.

Ans. { Courses 12° 55′ and 38° 45′
diff : lat 39 miles.

60. Two ships sail from the same place, one Sx°E, a miles, the other S2x°W, 2a miles, reaching the same parallel: find their courses.

Ans. Cos $x = \frac{1 + \sqrt{33}}{8}$; S 32° 29′ E, S 64 58 W

61. When the Sun's altitude was 2θ, the shadow of a perpendicular stick was a; and on the same day when his altitude was θ, the shadow was $a + b$; show that cos $2\theta = \frac{a}{b}$.

62. In lat 48°, the altitude of a body on the prime vertical was double its altitude on the six o'clock hour circle on the same day: find its declination, and the altitudes at each position.

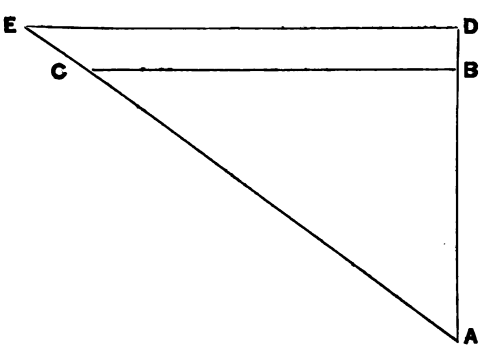
Ans. { On the six o'clock hour circle, 25° 6′
„ „ prime vertical 50 12
Declination 34 48

63. A Polacre Brig riding at anchor in a sea-way, pitched away her mainmast (a single stick), and as the broken part hung down, the mast-head touched the deck at 64 feet distant from the partners of the mast. The crew fished and scarphed the mast, and got it on end again, but when up, they found by the rigging that it was 7 feet shorter than before. The brig was afterwards run foul of, and her mast was carried away again in the same place, but this time, as the broken part hung down, the mast-head touched the deck at 55 feet distant from the part. Required to find the height of the mast at first, also, the height of the broken part, or stump, above the deck, supposing the mast to be perpendicular to the deck. Ans. 128, and 48 feet.

APPENDIX.

MERCATOR'S SAILING.

In "Plane Sailing," a right angled triangle is constructed similar to the numerous elementary triangles formed by a rhumb line cutting the meridians on the globe, and having its sides equal in length to the sum of the corresponding sides of the small triangles. Now, if these small triangles of the globe be projected on a Mercator's chart, and another similar triangle be constructed as in "plane sailing," with its sides equal in length to the sum of the corresponding sides of the projected triangles, we shall obtain a triangle similar to the first, or "plane sailing" triangle.



Let ABC, and ADE be the two similar triangles, having the common angle at A. Then AB is the difference of latitude: AD the meridional difference of latitude: BC the departure: DE the

difference of longitude: AC the distance sailed: AE the projected distance: and the angle A the course.

$$\text{Then } \frac{\text{ED}}{\text{DA}} = \tan \text{A}$$

$$i.e., \frac{\text{diff long}}{\text{mer d lat}} = \tan \text{course}$$

also,

$$\text{ED} : \text{DA} :: \text{CB} : \text{BA}$$

$$\text{D long} : \text{mer d lat} :: \text{dep} : \text{d lat}$$

therefore,

$$\text{d long} = \frac{\text{mer d lat} \times \text{dep}}{\text{d lat}}$$

MERCATOR'S CHART.

Imagine the Earth to be touched at every point of the equator by a circumscribed cylinder of white paper, and conceive the eye to be placed at the centre of the globe. If the meridians of the globe be delineated on the paper cylinder as they appear to the eye, they will become straight lines parallel to each other, and the parallels of latitude will be represented by circles equal to the equator.

Let the cylinder be now unrolled into a plane surface; the equator has now become a straight line; the meridians have become straight lines at right angles to it, and parallel to each other; and the parallels of latitude also straight lines everywhere equal to the equator.

Now, a chart to be of service to the navigator must represent the precise relation that exists between the minute divisions of the meridian, and the minute divisions of a parallel near to some point on the globe. Thus, if on the globe, the length of a minute of the parallel of 60° be one half the length of a minute of a

meridian, then, on a chart, that same relation should hold good. Or, if in lat 70°, the length of a minute of a meridian be about thrice the length of a minute of the parallel, the same relation must obtain on the chart.

But these relations the above projection does not accurately give, and recourse is had to a more exact method, still keeping to the idea of the developed cylinder.

On the globe, if a minute of the equator be taken as the limit of measurement, then, a minute of any parallel of latitude must be multiplied by the secant of that latitude, to be made equal to a minute of the equator. But as was shown above, the parallels have become equal to the equator (and therefore to the meridians), consequently they have become multiplied by the secants of their latitudes; and, in order that the proper relations may still be preserved at all points, the minutes of the meridians must be increased in the same degree, that is, in the ratio of the secants of the latitudes of those points.

The length, therefore, of a minute of a meridian at any particular point on a Mercator's chart, is found by multiplying a true minute of the meridian by the secant of the latitude of that point; and the sum of all the increased minutes contained in a portion of a meridian between two parallels is termed the "meridional difference of latitude." A table containing all these secants for every minute of latitude is called a table of "meridional parts."

It will be obvious that a chart or map constructed on such a framework will give an entirely false idea of the relative areas of countries in different latitudes. At the equator alone, portions of the globe will appear as they are on the globe, while the more distant a country is from the equator, the more is its area exaggerated; the shape is all that remains true. This disadvantage, however, is more than counterbalanced by the precious properties it possesses for the purposes of navigation.

The shortest distance between two points on the globe is along the arc of a great circle joining them; but, as an arc of a great circle cuts the meridians at different angles, to sail on it, it would be necessary to change the "course" continually.

The curve, however, which does cut the meridians at the same angle, and which, for distances not immense, is little longer than the arc of a great circle, is called a "rhumb line;" and, as only straight lines can cut parallel straight lines at the same angle, it follows that a rhumb line must be delineated on a Mercator's chart by a straight line.

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